

Generalized Fibonacci and Lucas cubes arising from powers of paths and cycles

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Abstract

The paper deals with some generalizations of Fibonacci and Lucas sequences, arising from powers of paths and cycles, respectively.

In the first part of the work we provide a formula for the number of edges of the Hasse diagram of the independent sets of the h^{th} power of a path ordered by inclusion. For $h = 1$ such a diagram is called a Fibonacci cube, and for $h > 1$ we obtain a generalization of the Fibonacci cube. Consequently, we derive a generalized notion of Fibonacci sequence, called h -Fibonacci sequence. Then, we show that the number of edges of a generalized Fibonacci cube is obtained by convolution of an h -Fibonacci sequence with itself.

In the second part we consider the case of cycles. We evaluate the number of edges of the Hasse diagram of the independent sets of the h^{th} power of a cycle ordered by inclusion. For $h = 1$ such a diagram is called Lucas cube, and for $h > 1$ we obtain a generalization of the Lucas cube. We derive then a generalized version of the Lucas sequence, called h -Lucas sequence. Finally, we show that the number of edges of a generalized Lucas cube is obtained by an appropriate convolution of an h -Fibonacci sequence with an h -Lucas sequence.

Keywords: independent set, path, cycle, power of graph, Fibonacci cube, Lucas cube, Fibonacci number, Lucas number,

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1. Introduction

For a graph \mathbf{G} we denote by $V(\mathbf{G})$ the set of its vertices, and by $E(\mathbf{G})$ the set of its edges.

Definition 1.1. For $n, h \geq 0$,

- (i) the h -power of a path, denoted by $\mathbf{P}_n^{(h)}$, is a graph with n vertices v_1, v_2, \dots, v_n such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{P}_n^{(h)})$ if and only if $|j - i| \leq h$;

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- (ii) the h -power of a cycle, denoted by $\mathbf{C}_n^{(h)}$, is a graph with n vertices v_1, v_2, \dots, v_n such that, for $1 \leq i, j \leq n, i \neq j, (v_i, v_j) \in E(\mathbf{C}_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$.

Thus, for instance, $\mathbf{P}_n^{(0)}$ and $\mathbf{C}_n^{(0)}$ are the graphs made of n isolated nodes, $\mathbf{P}_n^{(1)}$ is the path with n vertices, and $\mathbf{C}_n^{(1)}$ is the cycle with n vertices. Figure 1 shows some powers of paths and cycles.

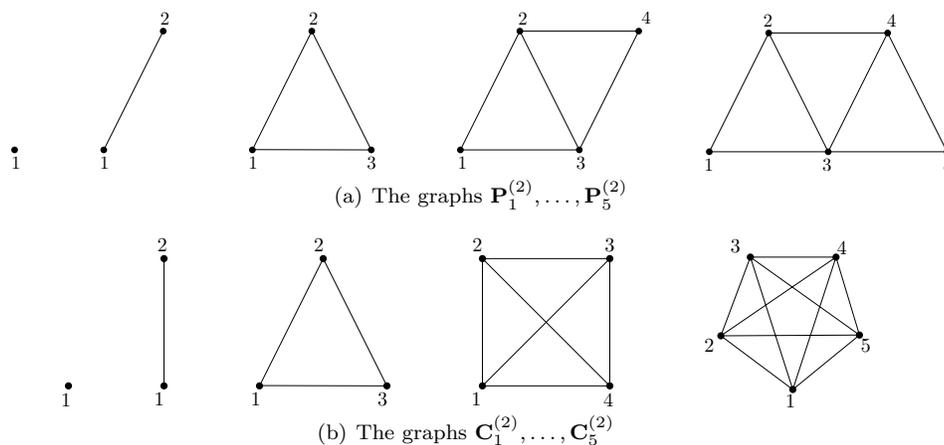


Figure 1: Some powers of paths and cycles.

Definition 1.2. An *independent set* of a graph \mathbf{G} is a subset of $V(\mathbf{G})$ not containing adjacent vertices.

Let $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ be the Hasse diagrams of the posets of independent sets of $\mathbf{P}_n^{(h)}$, and $\mathbf{C}_n^{(h)}$, respectively, ordered by inclusion. Clearly, $\mathbf{H}_n^{(0)} \cong \mathbf{M}_n^{(0)}$ is a Boolean lattice with n atoms (n -cube, for short).

Before introducing the main results of the paper, we now provide some background on Fibonacci and Lucas cubes. Every independent set S of $\mathbf{P}_n^{(h)}$ can be represented by a binary string $b_1 b_2 \cdots b_n$, where, for $i \in \{1, \dots, n\}$, $b_i = 1$ if and only if $v_i \in S$. Specifically, each independent set of $\mathbf{P}_n^{(h)}$ is associated with a binary string of length n such that the distance between any two 1's of the string is greater than h . Following [10] (see also [7]), a *Fibonacci string of order n* is a binary strings of length n without consecutive 1's. Recalling that the Hamming distance between two binary strings α and β is the number $H(\alpha, \beta)$ of bits where α and β differ, we can define the *Fibonacci cube of order n* , denoted Γ_n , as the graph (V, E) , where V is the set of all Fibonacci strings of order n and, for all $\alpha, \beta \in V$, $(\alpha, \beta) \in E$ if and only if $H(\alpha, \beta) = 1$. One can observe that for $h = 1$ the binary strings associated with independent sets of $\mathbf{P}_n^{(h)}$ are *Fibonacci strings of order n* , and the Hasse diagram of the set of all such strings ordered bitwise (*i.e.*, for $S = b_1 b_2 \cdots b_n$ and $T = c_1 c_2 \cdots c_n$, $S \geq T$ if and only

if $b_i \geq c_i$, for every $i \in \{1, \dots, n\}$) is Γ_n . Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [3], and their combinatorial structure has been further investigated, *e.g.* in [8, 10]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, *e.g.*, [4, 7]).

Remark. Consider the *generalized Fibonacci cubes* described in [4], *i.e.*, the graphs $\mathbf{B}_n(\alpha)$ obtained from the n -cube \mathbf{B}_n of all binary strings of length n by removing all vertices that contain the binary string α as a substring. In this notation the Fibonacci cube is $\mathbf{B}_n(11)$. It is not difficult to see that $\mathbf{H}_n^{(h)}$ cannot be expressed, in general, in terms of $\mathbf{B}_n(\alpha)$. Instead we have:

$\mathbf{H}_n^{(2)} = \mathbf{B}_n(11) \cap \mathbf{B}_n(101)$, $\mathbf{H}_n^{(3)} = \mathbf{B}_n(11) \cap \mathbf{B}_n(101) \cap \mathbf{B}_n(1001)$, \dots , where $\mathbf{B}_n(\alpha) \cap \mathbf{B}_n(\beta)$ is the subgraph of \mathbf{B}_n obtained by removing all strings that contain either α or β .

A similar argument can be carried out for cycles. Indeed, every independent set S of $\mathbf{C}_n^{(h)}$ can be represented by a circular binary string (*i.e.*, a sequence of 0's and 1's with the first and last bits considered to be adjacent) $b_1 b_2 \dots b_n$, where, for $i \in \{1, \dots, n\}$, $b_i = 1$ if and only if $v_i \in S$. Thus, each independent set of $\mathbf{C}_n^{(h)}$ is associated with a circular binary string of length n such that the distance between any two 1's of the string is greater than h . A *Lucas cube of order n* , denoted Λ_n , is defined as the graph whose vertices are the binary strings of length n without either two consecutive 1's or a 1 in the first and in the last position, and in which the vertices are adjacent when their Hamming distance is exactly 1 (see [9]). For $h = 1$ the Hasse diagram of the set of all circular binary strings associated with independent sets of $\mathbf{C}_n^{(h)}$ ordered bitwise is Λ_n . A generalization of the notion of Lucas cubes has been proposed in [5].

Remark. Consider the *generalized Lucas cubes* described in [5], that is, the graphs $\mathbf{B}_n(\hat{\alpha})$ obtained from the n -cube \mathbf{B}_n of all binary strings of length n by removing all vertices that have a *circular containing* α as a substring (*i.e.*, such that α is contained in the circular binary strings obtained by connecting first and last bits of the string). In this notation the Lucas cube is $\mathbf{B}_n(\widehat{11})$. It is not difficult to see that $\mathbf{M}_n^{(h)}$ cannot be expressed, in general, in terms of $\mathbf{B}_n(\hat{\alpha})$. Instead we have:

$$\mathbf{M}_n^{(2)} = \mathbf{B}_n(\widehat{11}) \cap \mathbf{B}_n(\widehat{101}), \mathbf{M}_n^{(3)} = \mathbf{B}_n(\widehat{11}) \cap \mathbf{B}_n(\widehat{101}) \cap \mathbf{B}_n(\widehat{1001}), \dots$$

To the best of our knowledge, our $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ are new generalizations of Fibonacci and Lucas cubes, respectively.

In the first part of this paper (which is an extended version of [1] — see remark at the end of this section) we evaluate $p_n^{(h)}$, *i.e.*, the number of *independent sets* of $\mathbf{P}_n^{(h)}$, and $H_n^{(h)}$, *i.e.*, the number of edges of $\mathbf{H}_n^{(h)}$. We then introduce a generalization of the Fibonacci sequence, that we call *h -Fibonacci sequence* and denote by $\mathcal{F}^{(h)}$. Such integer sequence is based on the values of $p_n^{(h)}$. Our main result (Theorem 3.4) is that, for $n, h \geq 0$, the sequence $H_n^{(h)}$ is obtained by convolving the sequence $\mathcal{F}^{(h)}$ with itself.

In the second part we deal with power of cycles, and derive similar results

for this case. Specifically, we compute $c_n^{(h)}$, *i.e.*, the number of *independent sets* of $\mathbf{C}_n^{(h)}$, and $M_n^{(h)}$, *i.e.*, the number of edges of $\mathbf{M}_n^{(h)}$. Further, we introduce a generalization of the Lucas sequence, that we call *h-Lucas sequence* and denote by $\mathcal{L}^{(h)}$. Such integer sequence is based on the values of $c_n^{(h)}$. The analogous of Theorem 3.4 in the Lucas case (Theorem 5.4) states that, for $n > h \geq 0$, the sequence $M_n^{(h)}$ is obtained by an appropriate convolution between the sequences $\mathcal{F}^{(h)}$ and $\mathcal{L}^{(h)}$.

Remark. As mentioned before, this work is an extended version of [1]. (The extended abstract [1] has been presented at the conference Combinatorics 2012, Perugia, Italy, 2012, and consequently published in ENDM.) Specifically, Section 2 of the present paper is [1, Section 2] enriched with some details and full proofs, while Section 3 is [1, Section 3], with corrected notation and some remarks added. Lemma 3.1, Lemma 3.2, and Theorem 3.4 have remained as they were in [1]: we decided to keep full proofs of these results, to make the content clearer and in order for the paper to be self-contained.

2. The independent sets of powers of paths

For $n, h, k \geq 0$, we denote by $p_{n,k}^{(h)}$ the number of independent k -subsets of $\mathbf{P}_n^{(h)}$.

Remark. $p_{n,k}^{(1)}$ counts the number of binary strings $\alpha \in \Gamma_n$ with k 1's.

Lemma 2.1. *For $n, h, k \geq 0$,*

$$p_{n,k}^{(h)} = \binom{n - hk + h}{k}.$$

This is Theorem 1 of [2]. An alternative proof follows.

Proof. By Definition 1.2, any two elements v_i, v_j of an independent set of $\mathbf{P}_n^{(h)}$ must satisfy $|j - i| > h$. It is straightforward to check that whenever $n - hk + h < 0$, $p_{n,k}^{(h)} = 0 = \binom{n - hk + h}{k}$. It is also immediate to see that when $n = h = 0$ our lemma holds true.

Suppose now $n - hk + h \geq 0$. We can complete the proof of our lemma by establishing a bijection between independent k -subset of $\mathbf{P}_n^{(h)}$ and k -subsets of a set with $(n - hk + h)$ elements. Let \mathcal{K} be the set of all k -subsets of a set $B = \{b_1, b_2, \dots, b_{n - hk + h}\}$, and \mathcal{I}_k the set of all independent k -subsets of $\mathbf{P}_n^{(h)}$. Consider the map $f : \mathcal{K} \rightarrow \mathcal{I}_k$ such that, for any $S = \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\} \in \mathcal{K}$, with $1 \leq i_1 < i_2 < \dots < i_k \leq n - hk + h$,

$$f(\{b_{i_1}, b_{i_2}, \dots, b_{i_j}, \dots, b_{i_k}\}) = \{v_{i_1}, v_{i_2+h}, \dots, v_{i_j+(j-1)h}, \dots, v_{i_k+(k-1)h}\}.$$

Claim 1. The map f associates an independent k -subset of $\mathbf{P}_n^{(h)}$ with each k -subset $S = \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\} \in \mathcal{K}$.

To see this we first remark that $f(S)$ is a k -subset of $V(\mathbf{P}_n^{(h)})$. Furthermore, for each pair $b_{i_j}, b_{i_{j+t}} \in S$, with $t > 0$, we have

$$i_{j+t} + (j+t-1)h - (i_j + (j-1)h) = i_{j+t} - i_j + th > h.$$

Hence, by Definition 1.1, $(f(b_{i_j}), f(b_{i_{j+t}})) = (v_{i_j+(j-1)h}, v_{i_{j+t}+(j+t-1)h}) \notin E(\mathbf{P}_n^{(h)})$. Thus, $f(S)$ is an independent set of $\mathbf{P}_n^{(h)}$.

Claim 2. The map f is bijective.

It is easy to see that f is injective. Then, we consider the map $f^{-1} : \mathcal{I}_k \rightarrow \mathcal{K}$ such that, for any $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \in \mathcal{I}$, with $1 \leq i_1 < i_2 < \dots < i_k \leq n$,

$$f^{-1}(\{v_{i_1}, v_{i_2}, \dots, v_{i_j}, \dots, v_{i_k}\}) = \{b_{i_1}, b_{i_2-h}, \dots, b_{i_j-(j-1)h}, \dots, b_{i_k-(k-1)h}\}.$$

Following the same steps as for f , one checks that f^{-1} is injective. Thus, f is surjective.

We have established a bijection between independent k -subsets of $\mathbf{P}_n^{(h)}$ and k -subsets of a set with $(n - hk + h) \geq 0$ elements. The lemma is proved. \square

Some values of $p_{n,k}^{(h)}$ are shown in Tables 1–3 (Section 6). The coefficients $p_{n,k}^{(h)}$ also enjoy the following property: $p_{n,k}^{(h)} = p_{n-k+1,k}^{(h-1)}$.

For $n, h \geq 0$, the number of independent sets of $\mathbf{P}_n^{(h)}$ is

$$p_n^{(h)} = \sum_{k \geq 0} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n - hk + h}{k}.$$

Remark. Denote by F_n the n^{th} element of the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$ is the number of vertices of the Fibonacci cube of order n .

The following, simple fact is crucial for our work.

Lemma 2.2. For $n, h \geq 0$,

$$p_n^{(h)} = \begin{cases} n + 1 & \text{if } n \leq h + 1, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h + 1. \end{cases}$$

A proof of this Lemma can also be obtained using the first part of [2, Proof of Theorem 1].

Proof. For $n \leq h + 1$, by Definition 1.2, the independent sets of $\mathbf{P}_n^{(h)}$ have no more than 1 element. Thus, there are $n + 1$ independent sets of $\mathbf{P}_n^{(h)}$.

Consider the case $n > h + 1$. Let \mathcal{I} be the set of all independent sets of $\mathbf{P}_n^{(h)}$, let \mathcal{I}_{in} be the set of the independent sets of $\mathbf{P}_n^{(h)}$ that contain v_n , and let $\mathcal{I}_{out} = \mathcal{I} \setminus \mathcal{I}_{in}$. The elements of \mathcal{I}_{out} are in one-to-one correspondence with the $p_{n-1}^{(h)}$ independent sets of $\mathbf{P}_{n-1}^{(h)}$, and those of \mathcal{I}_{in} are in one-to-one correspondence with the $p_{n-h-1}^{(h)}$ independent sets of $\mathbf{P}_{n-h-1}^{(h)}$. \square

Tables 4 displays a few values of $p_n^{(h)}$.

3. Generalized Fibonacci numbers and generalized Fibonacci cubes

Figure 2 shows a few Hasse diagrams $\mathbf{H}_n^{(h)}$. Notice that, as stated in the introduction, for each n , $\mathbf{H}_n^{(1)}$ is the Fibonacci cube Γ_n .

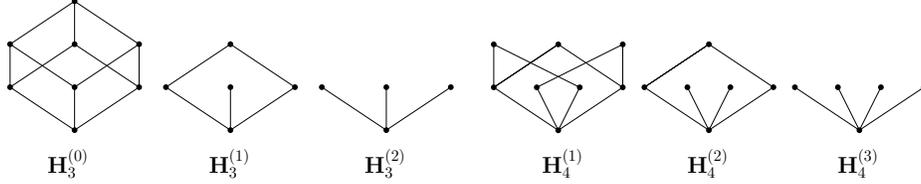


Figure 2: Some $\mathbf{H}_n^{(h)}$.

Let $H_n^{(h)}$ be the number of edges of $\mathbf{H}_n^{(h)}$. Noting that in $\mathbf{H}_n^{(h)}$ each non-empty independent k -subset covers exactly k independent $(k-1)$ -subsets, we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k p_{n,k}^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \binom{n-hk+h}{k}. \quad (1)$$

Remark. $H_n^{(1)}$ counts the number of edges of Γ_n .

Let now $T_{k,i}^{(n,h)}$ be the number of independent k -subsets of $\mathbf{P}_n^{(h)}$ containing the vertex v_i , and let, for $h, k \geq 0$, $n \in \mathbb{Z}$, $\bar{p}_{n,k}^{(h)} = \begin{cases} p_{0,k}^{(h)} & \text{if } n < 0, \\ p_{n,k}^{(h)} & \text{if } n \geq 0. \end{cases}$

Lemma 3.1. For $n, h, k \geq 0$, and $1 \leq i \leq n$,

$$T_{k,i}^{(n,h)} = \sum_{r=0}^{k-1} \bar{p}_{i-h-1,r}^{(h)} \bar{p}_{n-i-h,k-1-r}^{(h)}.$$

Proof. No independent set of $\mathbf{P}_n^{(h)}$ containing v_i contains any of the elements $v_{i-h}, \dots, v_{i-1}, v_{i+1}, \dots, v_{i+h}$. Let r and s be non-negative integers whose sum is $k-1$. Each independent k -subset of $\mathbf{P}_n^{(h)}$ containing v_i can be obtained by adding v_i to a $(k-1)$ -subset $R \cup S$ such that

- (a) $R \subseteq \{v_1, \dots, v_{i-h-1}\}$ is an independent r -subset of $\mathbf{P}_n^{(h)}$;
- (b) $S \subseteq \{v_{i+h+1}, \dots, v_n\}$ is an independent s -subset of $\mathbf{P}_n^{(h)}$.

Viceversa, one can obtain each of this pairs of subsets by removing v_i from an independent k -subset of $\mathbf{P}_n^{(h)}$ containing v_i . Thus, $T_{k,i}^{(n,h)}$ is obtained by counting independently the subsets of type (a) and (b). The remark that the subsets of type (b) are in bijection with the independent s -subsets of $\mathbf{P}_{n-i-h}^{(h)}$ proves the lemma. \square

Remark. $T_{k,i}^{(n,1)}$ counts the number of strings $\alpha = b_1 b_2 \cdots b_n \in \Gamma_n$ such that: (i) $H(\alpha, 00 \cdots 0) = k$, and (ii) $b_i = 1$.

In order to obtain our main result, we prepare a lemma.

Lemma 3.2. *For positive n ,*

$$\sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^n T_{k,i}^{(n,h)} = H_n^{(h)}.$$

Proof. The inner sum counts the number of k -subsets exactly k times, one for each element of the subset. That is, $\sum_{i=1}^n T_{k,i}^{(n,h)} = k p_{n,k}^{(h)}$. Hence the lemma follows directly from Equation (1). \square

Next we introduce a family of Fibonacci-like sequences.

Definition 3.3. For $h \geq 0$, and $n \geq 1$, we define the h -Fibonacci sequence $\mathcal{F}^{(h)} = \{F_n^{(h)}\}_{n \geq 1}$ whose elements are

$$F_n^{(h)} = \begin{cases} 1 & \text{if } n \leq h+1, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h+1. \end{cases}$$

The first values of the h -Fibonacci sequences, for $h \in \{1, \dots, 10\}$, are shown in Table 5. From Lemma 2.2, and setting for $h \geq 0$, and $n \in \mathbb{Z}$, $\bar{p}_n^{(h)} = \begin{cases} p_0^{(h)} & \text{if } n < 0, \\ p_n^{(h)} & \text{if } n \geq 0, \end{cases}$ we have that,

$$F_i^{(h)} = \bar{p}_{i-h-1}^{(h)}, \text{ for each } i \geq 1. \quad (2)$$

Thus, our Fibonacci-like sequences are obtained by prepending h 1's to the sequence $p_0^{(h)}, p_1^{(h)}, \dots$. Therefore, we have:

- $\mathcal{F}^{(0)} = 1, 2, 4, \dots, 2^n, \dots$;
- $\mathcal{F}^{(1)}$ is the Fibonacci sequence;
- more generally, $\mathcal{F}^{(h)} = \underbrace{1, \dots, 1}_h, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \dots$

In the following, we define the discrete convolution operation $*$, as follows.

$$\left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)} \right) (n) \doteq \sum_{i=1}^n F_i^{(h)} F_{n-i+1}^{(h)} \quad (3)$$

Theorem 3.4. *For $n, h \geq 0$, the following holds*

$$H_n^{(h)} = \left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)} \right) (n).$$

Proof. The sum $\sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)}$ counts the number of independent sets of $\mathbf{P}_n^{(k)}$ containing v_i . We can also obtain such a value by counting the independent sets of both $\{v_1, \dots, v_{i-h-1}\}$, and $\{v_{i+h+1}, \dots, v_n\}$. Thus, we have:

$$\sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.$$

Using Lemma 3.2 we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^n T_{k,i}^{(n,h)} = \sum_{i=1}^n \sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \sum_{i=1}^n \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.$$

By Equation (2) we have $\sum_{i=1}^n \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} = \sum_{i=1}^n F_i^{(h)} F_{n-i+1}^{(h)}$. By (3), the theorem is proved. \square

We display some values of $H_n^{(h)}$ in Table 6 (Section 6).

Remark. For $h = 1$, we obtain the number of edges of Γ_n by using Fibonacci numbers:

$$H_n^{(1)} = \sum_{i=1}^n F_i F_{n-i+1}.$$

The latter result is [6, Proposition 3].

4. The independent sets of powers of cycles

For $n, h, k \geq 0$, we denote by $c_{n,k}^{(h)}$ the number of independent k -subsets of $\mathbf{C}_n^{(h)}$.

Remark. For $n > 1$, $c_{n,k}^{(1)}$ counts the number of binary strings $\alpha \in \Lambda_n$ with k 1's.

Lemma 4.1. For $n, h \geq 0$, and $k > 1$,

$$c_{n,k}^{(h)} = \frac{n}{k} \binom{n-hk-1}{k-1}.$$

Moreover, $c_{n,0}^{(h)} = 1$, and $c_{n,1}^{(h)} = n$, for each $n, h \geq 0$.

Proof. Fix an element $v_i \in V(\mathbf{C}_n^{(h)})$, and let $n > 2h$. Any independent set of $\mathbf{C}_n^{(h)}$ containing v_i does not contain the h elements preceding v_i and the h elements following v_i . Thus, the number of independent k -subsets of $\mathbf{C}_n^{(h)}$ containing v_i equals

$$p_{n-2h-1,k-1}^{(h)} = \binom{n-hk-1}{k-1}.$$

The total number of independent k -subsets of $\mathbf{C}_n^{(h)}$ is obtained by multiplying $p_{n-2h-1,k-1}^{(h)}$ by n , then dividing it by k (each subset is counted k times by the previous proceeding). The case $n \leq 2h$, as well as the cases $k = 0, 1$, can be easily verified. \square

Some values of $c_{n,k}^{(h)}$ are displayed in Tables 7–9.

For $n, h \geq 0$, the number of all independent sets of $\mathbf{C}_n^{(h)}$ is

$$c_n^{(h)} = \sum_{k \geq 0} c_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} c_{n,k}^{(h)}, \quad (4)$$

Remark. Denote by L_n the n^{th} element of the Lucas sequence $L_1 = 1$, $L_2 = 3$, and $L_i = L_{i-1} + L_{i-2}$, for $i > 2$. Then, for $n > 1$, $c_n^{(1)} = L_n$ is the number of elements of the Lucas cube of order n .

Some values of $c_n^{(h)}$ are shown in Table 10. The coefficients $c_n^{(h)}$ satisfy a recursion that closely resembles that of Lemma 2.2.

Lemma 4.2. For $n, h \geq 0$,

$$c_n^{(h)} = \begin{cases} n+1 & \text{if } n \leq 2h+1, \\ c_{n-1}^{(h)} + c_{n-h-1}^{(h)} & \text{if } n > 2h+1. \end{cases} \quad (5)$$

Proof. The case $n \leq 2h+1$ can be easily checked. The case $2h+1 < n \leq 3h+2$ is discussed at the end of this proof. Let $n > 3h+2$, and let \mathcal{I} be the set of the independent sets of $\mathbf{C}_n^{(h)}$. Let \mathcal{I}_{in} be the subset of these sets that (i) do not contain v_n , and that (ii) do not contain any of the following pairs: $(v_1, v_{n-h}), (v_2, v_{n-h+1}), \dots, (v_h, v_{n-1})$. Let then \mathcal{I}_{out} be the subset of the remaining independent sets of $\mathbf{C}_n^{(h)}$.

It is easy to see that the elements of \mathcal{I}_{in} are exactly the independent sets of $\mathbf{C}_{n-1}^{(h)}$. Indeed, v_n is not a vertex of $\mathbf{C}_{n-1}^{(h)}$ and the vertices of pairs $(v_1, v_{n-h}), (v_2, v_{n-h+1}), \dots, (v_h, v_{n-1})$ are adjacent in $\mathbf{C}_{n-1}^{(h)}$. On the other hand, to show that

$$|\mathcal{I}_{out}| = c_{n-h-1}^{(h)}$$

we argue as follows. First we recall (see the proof of Lemma 4.1) that the number of independent k -subsets of $\mathbf{C}_n^{(h)}$ that contain v_n is $p_{n-2h-1, k-1}^{(h)}$. Secondly we claim that the number of independent k -subsets of $\mathbf{C}_n^{(h)}$ containing one of the pairs $(v_1, v_{n-h}), (v_2, v_{n-h+1}), \dots, (v_h, v_{n-1})$ is $h p_{n-3h-2, k-2}^{(h)}$. To see this, consider the pair (v_1, v_{n-h}) . The independent sets containing such a pair do not contain the h vertices from v_{n-h+1} to v_n , do not contain the h vertices from v_2 to v_{h+1} , and do not contain the h vertices from v_{n-2h} to v_{n-h-1} . Thus, the removal of such vertices and of the vertices v_1 and v_{n-h} turns $\mathbf{C}_n^{(h)}$ into $\mathbf{P}_{n-3h-2}^{(h)}$. Hence we can obtain all the independent k -subsets of $\mathbf{C}_n^{(h)}$ that contain the pair (v_1, v_{n-h}) by simply adding these two vertices to one of the $p_{n-3h-2, k-2}^{(h)}$ independent $k-2$ -subsets of $\mathbf{P}_{n-3h-2}^{(h)}$. Same reasoning can be carried out for any other one of the pairs: $(v_2, v_{n-h+1}), \dots, (v_h, v_{n-1})$.

Using Lemmas 2.1 and 4.1 one can easily derive that

$$p_{n-2h-1, k-1}^{(h)} + h p_{n-3h-2, k-2}^{(h)} = c_{n-h-1, k-1}^{(h)}.$$

Hence, we derive the size of \mathcal{I}_{out} :

$$|\mathcal{I}_{out}| = c_{n-h-1}^{(h)} = \sum_{k \geq 1} p_{n-2h-1, k-1}^{(h)} + h \sum_{k \geq 2} p_{n-3h-2, k-2}^{(h)}.$$

Summing up we have shown that $|\mathcal{I}| = |\mathcal{I}_{in}| + |\mathcal{I}_{out}|$, that is

$$c_n^{(h)} = c_{n-1}^{(h)} + c_{n-h-1}^{(h)}.$$

The proof of the case $2h + 1 < n \leq 3h + 2$ is obtained in a similar way, observing that $|\mathcal{I}_{out}| = n - h$, and that $n - h - 1 \leq 2h + 1$. \square

5. Generalized Lucas cubes and Lucas numbers

Figure 3 shows a few Hasse diagrams $\mathbf{M}_n^{(h)}$. Notice that, as stated in the introduction, for each $n \geq 1$, $\mathbf{M}_n^{(1)}$ is the Lucas cube Λ_n .

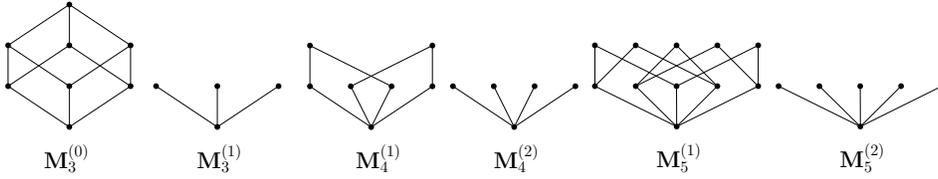


Figure 3: Some $\mathbf{M}_n^{(h)}$.

Let $M_n^{(h)}$ be the number of edges of $\mathbf{M}_n^{(h)}$. As done in Section 3 for the case of paths, we immediately provide a formula for $M_n^{(h)}$:

$$M_n^{(h)} = \sum_{k=0}^{\lceil n/h+1 \rceil} k c_{n,k}^{(h)} = n \sum_{k=0}^{\lceil n/h+1 \rceil} \binom{n-hk-1}{k-1}. \quad (6)$$

Remark. For $n > 1$, $M_n^{(1)}$ counts the number of edges of Λ_n . As shown in [9, Proposition 4(ii)], $M_n^{(1)} = nF_{n-1}$.

As shown in the proof of Lemma 4.1, the value

$$p_{n-2h-1, k-1}^{(h)} = \binom{n-hk-1}{k-1}$$

is the analogue of the coefficient $T_{k,i}^{(n,h)}$: in the case of cycles we have no dependencies on i , because each choice of vertex is equivalent. We can obtain $M_n^{(h)}$ in terms of a Fibonacci-like sequence, as follows.

Proposition 5.1. *For $n > h \geq 0$, the following holds*

$$M_n^{(h)} = nF_{n-h}^{(h)}.$$

Proof. Using Equation (2) we obtain:

$$M_n^{(h)} = n \sum_{k=1}^{\lceil n/(h+1) \rceil} \bar{p}_{n-2h-1, k-1}^{(h)} = n \bar{p}_{n-2h-1}^{(h)} = n F_{n-h}^{(h)}.$$

□

In analogy with Section 3, we introduce a family of Lucas-like sequences.

Definition 5.2. For $h \geq 0$, and $n \geq 1$, we define the h -Lucas sequence $\mathcal{L}^{(h)} = \{L_n^{(h)}\}_{n \geq 1}$ whose elements are

$$L_n^{(h)} = \begin{cases} h+1 & \text{if } n = 1, \\ 1 & \text{if } 2 \leq n \leq h+1, \\ L_{n-1}^{(h)} + L_{n-h-1}^{(h)} & \text{if } n > h+1. \end{cases}$$

The first values of the h -Lucas sequences, for $h \in \{0, \dots, 10\}$, are displayed in Table 11. We have that,

$$L_i^{(h)} = c_{i-1}^{(h)}, \quad \text{for each } i > h+1. \quad (7)$$

To prove the main result of this section, the following lemma is needed.

Lemma 5.3. For $n > h \geq 0$, the following holds

$$L_{n+1}^{(h)} = F_n^{(h)} + (h+1)F_{n-h}^{(h)}$$

Proof. The result is proved by induction on n . Indeed, $L_{n+1}^{(h)} = L_n^{(h)} + L_{n-h}^{(h)}$. Applying the inductive hypothesis, we have

$$L_{n+1}^{(h)} = F_{n-1}^{(h)} + (h+1)F_{n-h-1}^{(h)} + F_{n-h-1}^{(h)} + (h+1)F_{n-2h-1}^{(h)} = F_n^{(h)} + (h+1)F_{n-h}^{(h)}.$$

□

Finally, our analogous of Theorem 3.4, for cycles, is the following.

Theorem 5.4. For $n > h \geq 0$, the following holds

$$M_n^{(h)} = \left(\mathcal{F}^{(h)} * \mathcal{L}^{(h)} \right) (n-h).$$

Proof. By Proposition 5.1, the statement of the Theorem is equivalent to

$$\sum_{i=1}^{n-h} F_i^{(h)} L_{n-h+1-i}^{(h)} = n F_{n-h}^{(h)}. \quad (8)$$

Let $h = 0$. We have

$$\sum_{i=1}^n F_i^{(0)} L_{n+1-i}^{(0)} = \sum_{i=1}^n 2^{i-1} 2^{n-i} = \sum_{i=1}^n 2^{n-1} = n 2^{n-1} = n F_n^{(0)}.$$

Let $h = 1$. In this case the statement of the theorem reduces to the well known identity involving (classical) Fibonacci and Lucas sequences:

$$\sum_{i=1}^n F_i L_{n-i+1} = (n+1)F_n.$$

Let $h \geq 2$. We prove (8) by induction on n . If $n = h + 1$, then

$$\sum_{i=1}^{n-h} F_i^{(h)} L_{n-h+1-i}^{(h)} = F_1^{(h)} L_1^{(h)} = h + 1 = nF_1^{(h)}.$$

Let $\bar{n} > h + 1$, and suppose (inductive hypothesis) that (8) holds for every $1 < n \leq \bar{n}$. Let $m = \bar{n} - h$ (and note that $m \geq 2$). We need to prove that

$$\sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} = (m+h+1)F_{m+1}^{(h)}. \quad (9)$$

We find it convenient to define, for $h \geq 2$, the following integer sequences, which extend $F_n^{(h)}$ and $L_n^{(h)}$ to a range of negative integers.

$$\bar{F}_n^{(h)} = \begin{cases} 1 & \text{if } n = -h, \\ 0 & \text{if } -h < n \leq 0, \\ \bar{F}_{n-1}^{(h)} + \bar{F}_{n-h-1}^{(h)} & \text{if } n > 0. \end{cases} \quad (10)$$

$$\bar{L}_n^{(h)} = \begin{cases} h+1 & \text{if } n = -h, \\ -h & \text{if } n = -h+1, \\ 0 & \text{if } -h+1 < n \leq 0, \\ \bar{L}_{n-1}^{(h)} + \bar{L}_{n-h-1}^{(h)} & \text{if } n > 0. \end{cases} \quad (11)$$

Using Definitions 3.3 and 5.2 one can easily check that, for $n > 0$, $F_n^{(h)} = \bar{F}_n^{(h)}$, and $L_n^{(h)} = \bar{L}_n^{(h)}$. Hence, applying the recurrences in (10) and (11), we obtain

$$\begin{aligned} \sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} &= \sum_{i=1}^{m+1} \bar{F}_i^{(h)} \bar{L}_{m+2-i}^{(h)} = \\ &= \sum_{i=1}^{m+1} \bar{F}_{i-1}^{(h)} \bar{L}_{m+1-i}^{(h)} + \end{aligned} \quad (12)$$

$$+ \sum_{i=1}^{m+1} \bar{F}_{i-1}^{(h)} \bar{L}_{m+1-h-i}^{(h)} + \quad (13)$$

$$+ \sum_{i=1}^{m+1} \bar{F}_{i-h-1}^{(h)} \bar{L}_{m+1-i}^{(h)} + \quad (14)$$

$$+ \sum_{i=1}^{m+1} \bar{F}_{i-h-1}^{(h)} \bar{L}_{m+1-h-i}^{(h)}. \quad (15)$$

We compute the sums (12)–(15) separately.

Direct computation shows that (12) equals

$$\sum_{i=1}^{m-1} F_i^{(h)} L_{m-i}^{(h)} + \bar{F}_0^{(h)} \bar{L}_m^{(h)} + \bar{F}_m^{(h)} \bar{L}_0^{(h)}. \quad (16)$$

Applying the inductive hypotheses to the first term of (16), and observing that $\bar{L}_0^{(h)} = \bar{F}_0^{(h)} = 0$, we obtain that the sum (12) is

$$(m+h-1)F_{m-1}^{(h)}. \quad (17)$$

In order to compute (13) we distinguish two cases. If $m > h+1$, direct computation shows that (13) equals

$$\sum_{i=1}^{m-h-1} F_i^{(h)} L_{m-h-i}^{(h)} + \bar{F}_0^{(h)} \bar{L}_{m-h}^{(h)} + \bar{F}_{m-h}^{(h)} \bar{L}_0^{(h)} + \bar{F}_{m-h+1}^{(h)} \bar{L}_{-1}^{(h)} + \cdots + \bar{F}_m^{(h)} \bar{L}_{-h}^{(h)}. \quad (18)$$

We note that $m > h+1 \geq 3$. Applying the inductive hypotheses to the first term of (18), and observing that all other terms are zeroes, except $\bar{F}_m^{(h)} \bar{L}_{-h}^{(h)} = (h+1)F_m^{(h)}$, and $\bar{F}_{m-1}^{(h)} \bar{L}_{-h+1}^{(h)} = -hF_{m-1}^{(h)}$, we obtain that (13) equals

$$(m-1)F_{m-h-1}^{(h)} + (h+1)F_m^{(h)} - hF_{m-1}^{(h)}. \quad (\text{for } m > h+1) \quad (19)$$

If, on the other hand, $m \leq h+1$, we observe that the summands of (13) vanish, with the exception of $\bar{F}_m^{(h)} \bar{L}_{-h}^{(h)} = (h+1)F_m^{(h)}$, and $\bar{F}_{m-1}^{(h)} \bar{L}_{-h+1}^{(h)} = -hF_{m-1}^{(h)}$. Indeed, $m \geq 2$. Thus, the sum (13) is

$$(h+1)F_m^{(h)} - hF_{m-1}^{(h)}. \quad (\text{for } m \leq h+1) \quad (20)$$

A similar argument shows that (14) equals

$$(m-1)F_{m-h-1}^{(h)} + L_m^{(h)}, \quad (\text{for } m > h+1) \quad (21)$$

$$L_m^{(h)}. \quad (\text{for } m \leq h+1) \quad (22)$$

To calculate (15) we distinguish the four cases $m \leq h$, $m = h+1$, $h+1 < m \leq 2h+1$, and $m > 2h+1$. If $m > 2h+1$, (15) equals

$$\sum_{i=1}^{m-2h-1} F_i^{(h)} L_{m-2h-i}^{(h)} + \bar{L}_{-h}^{(h)} \bar{F}_{m-h}^{(h)} + \cdots + \bar{L}_0^{(h)} \bar{F}_{m-2h}^{(h)} + \bar{F}_{-h}^{(h)} \bar{L}_{m-h}^{(h)} + \cdots + \bar{F}_0^{(h)} \bar{L}_{m-2h}^{(h)}.$$

We note that $m-h-1 > h > 0$. Applying the inductive hypotheses to the first term of the preceding sum, and observing that all other terms are

zeroes, except $\bar{L}_{-h}^{(h)}\bar{F}_{m-h}^{(h)} = (h+1)F_{m-h}^{(h)}$, $\bar{L}_{-h+1}^{(h)}\bar{F}_{m-h-1}^{(h)} = -hF_{m-h-1}^{(h)}$, and $\bar{F}_{-h}^{(h)}\bar{L}_{m-h}^{(h)} = L_{m-h}^{(h)}$, we obtain that (15) is

$$(m-h-1)F_{m-2h-1}^{(h)} + (h+1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}^{(h)} \quad (\text{for } m > 2h+1) \quad (23)$$

To tackle the other cases, we expand the sum (15), and we check how the non-zeroes terms change depending on the constraints on m . We obtain that (15) equals

$$(h+1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}^{(h)}, \quad (\text{for } h+1 < m \leq 2h+1) \quad (24)$$

$$(h+1)F_{m-h}^{(h)} + L_{m-h}^{(h)}, \quad (\text{for } m = h+1) \quad (25)$$

$$0. \quad (\text{for } m \leq h) \quad (26)$$

Finally, we add up the four summands (12)–(15), distinguishing the four identified cases. If $m > 2h+1$, we sum (17), (19), (21), and (23), obtaining

$$\begin{aligned} \sum_{i=1}^{m+1} F_i^{(h)} L_{m+1-i}^{(h)} &= (m+h-1)F_{m-1}^{(h)} + 2(m-1)F_{m-h-1}^{(h)} + (h+1)F_m^{(h)} - hF_{m-1}^{(h)} + \\ &+ L_m^{(h)} + (m-h-1)F_{m-2h-1}^{(h)} + (h+1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}^{(h)}. \end{aligned}$$

Repeatedly applying the recurrences $F_i^{(h)} = F_{i-1}^{(h)} + F_{i-h-1}^{(h)}$ and $L_i^{(h)} = L_{i-1}^{(h)} + L_{i-h-1}^{(h)}$, and using Lemma 5.3, we obtain

$$\sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} = (m+h+1)F_{m+1}^{(h)},$$

which proves (8) in the case $m > 2h+1$.

If $h+1 < m \leq 2h+1$, we sum (17), (19), (21), and (24), and we obtain

$$\begin{aligned} \sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} &= (m+h-1)F_{m-1}^{(h)} + 2(m-1)F_{m-h-1}^{(h)} + (h+1)F_m^{(h)} - hF_{m-1}^{(h)} + \\ &+ L_m^{(h)} + (h+1)F_{m-h}^{(h)} - hF_{m-h-1}^{(h)} + L_{m-h}^{(h)}. \end{aligned}$$

Note that, by Definition 3.3, we have $F_{h+k}^{(h)} = k$, for $k \in \{1, h+2\}$. Moreover, $F_{m-h-1}^{(h)} = F_{m-h}^{(h)} = 1$, and, by Definition 5.2, $L_{m-h}^{(h)} = 1$. Applying Lemma 5.3 to $L_m^{(h)}$, we obtain

$$\begin{aligned} \sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} &= (h+1)F_m^{(h)} + mF_{m-1}^{(h)} + h+1 + 2m = \\ &= (h+1)(m-h) + m(m-1-h) + h+1 + 2m = \\ &= (m+h+1)(m+1-h) = (m+h+1)F_{m+1}^{(h)}. \end{aligned}$$

If $m = h + 1$, we sum (17), (20), (22), and (25), obtaining

$$\begin{aligned} \sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} &= (m + h - 1)F_{m-1}^{(h)} + (h + 1)F_m^{(h)} - hF_{m-1}^{(h)} + \\ &\quad + L_m^{(h)} + (h + 1)F_{m-h}^{(h)} + L_{m-h}^{(h)}. \end{aligned}$$

Noting that, by Definition 3.3 and Definition 5.2, $F_m^{(h)} = F_{m-1}^{(h)} = \dots = F_1^{(h)} = 1$, $L_m^{(h)} = 1$, and $L_{m-h}^{(h)} = h + 1$, we have

$$\sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} = 4h + 4 = (m + h + 1)2 = (m + h + 1)F_{m+1}^{(h)}.$$

In the last case, if $m < h + 1$, we sum (17), (20), (22), and (26), and we have

$$\sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} = (m + h - 1)F_{m-1}^{(h)} + (h + 1)F_m^{(h)} - hF_{m-1}^{(h)} + L_m^{(h)}.$$

Proceeding as in the previous case, and recalling that $m \geq 2$, we obtain

$$\sum_{i=1}^{m+1} F_i^{(h)} L_{m+2-i}^{(h)} = m + h - 1 + h + 1 - h + 1 = (m + h + 1)F_{m+1}^{(h)}.$$

In all cases we obtain the desired result, and the proof is complete. \square

We display some values of $M_n^{(h)}$ in Table 12.

6. Tables

We collect here some values obtained by computing the formula presented in the preceding sections.

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
k=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91
3	0	0	0	0	0	1	4	10	20	35	56	84	120	165	220	286
4	0	0	0	0	0	0	0	1	5	15	35	70	126	210	330	495
5	0	0	0	0	0	0	0	0	0	1	6	21	56	126	252	462
6	0	0	0	0	0	0	0	0	0	0	1	7	28	84	210	
7	0	0	0	0	0	0	0	0	0	0	0	0	1	8	36	
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	

Table 1: The number $p_{n,k}^{(1)}$ of independent k -subsets of $\mathbf{P}_n^{(1)}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
k=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	0	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91
3	0	0	0	0	0	0	1	4	10	20	35	56	84	120	165	220	
4	0	0	0	0	0	0	0	0	0	1	5	15	35	70	126	210	
5	0	0	0	0	0	0	0	0	0	0	0	0	1	6	21	56	
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 2: The number $p_{n,k}^{(2)}$ of independent k -subsets of $\mathbf{P}_n^{(2)}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
k=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	0	0	0	0	0	1	3	6	10	15	21	28	36	45	55	66	78	91
3	0	0	0	0	0	0	0	0	1	4	10	20	35	56	84	120	165	
4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	5	15	35	70
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 3: The number $p_{n,k}^{(3)}$ of independent k -subsets of $\mathbf{P}_n^{(3)}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
2	1	2	3	4	6	9	13	19	28	41	60	88	129	189
3	1	2	3	4	5	7	10	14	19	26	36	50	69	95
4	1	2	3	4	5	6	8	11	15	20	26	34	45	60
5	1	2	3	4	5	6	7	9	12	16	21	27	34	43
6	1	2	3	4	5	6	7	8	10	13	17	22	28	35
7	1	2	3	4	5	6	7	8	9	11	14	18	23	29
8	1	2	3	4	5	6	7	8	9	10	12	15	19	24
9	1	2	3	4	5	6	7	8	9	10	11	13	16	20
10	1	2	3	4	5	6	7	8	9	10	11	12	14	17

Table 4: The number $p_n^{(h)}$ of all independent sets of $\mathbf{P}_n^{(h)}$

	n=1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384
1	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
2	1	1	1	2	3	4	6	9	13	19	28	41	60	88	129
3	1	1	1	1	2	3	4	5	7	10	14	19	26	36	50
4	1	1	1	1	1	2	3	4	5	6	8	11	15	20	26
5	1	1	1	1	1	1	2	3	4	5	6	7	9	12	16
6	1	1	1	1	1	1	1	2	3	4	5	6	7	8	10
7	1	1	1	1	1	1	1	1	2	3	4	5	6	7	8
8	1	1	1	1	1	1	1	1	1	2	3	4	5	6	7
9	1	1	1	1	1	1	1	1	1	1	2	3	4	5	6
10	1	1	1	1	1	1	1	1	1	1	1	2	3	4	5

Table 5: Values of the h -Fibonacci sequence $\mathcal{F}^{(h)} = \{F_n^{(h)}\}_{n \geq 1}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13
h=0	0	1	4	12	32	80	192	448	1024	2304	5120	11264	24576	53248
1	0	1	2	5	10	20	38	71	130	235	420	744	1308	2285
2	0	1	2	3	6	11	18	30	50	81	130	208	330	520
3	0	1	2	3	4	7	12	19	28	42	64	97	144	212
4	0	1	2	3	4	5	8	13	20	29	40	56	80	115
5	0	1	2	3	4	5	6	9	14	21	30	41	54	72
6	0	1	2	3	4	5	6	7	10	15	22	31	42	55
7	0	1	2	3	4	5	6	7	8	11	16	23	32	43
8	0	1	2	3	4	5	6	7	8	9	12	17	24	33
9	0	1	2	3	4	5	6	7	8	9	10	13	18	25
10	0	1	2	3	4	5	6	7	8	9	10	11	14	19

Table 6: The number $H_n^{(h)}$ of edges of $\mathbf{H}_n^{(h)}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
k=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	0	0	0	2	5	9	14	20	27	35	44	54	65	77	90	104
3	0	0	0	0	0	0	2	7	16	30	50	77	112	156	210	275	352
4	0	0	0	0	0	0	0	2	9	25	55	105	182	294	450	660	
5	0	0	0	0	0	0	0	0	0	2	11	36	91	196	378	672	
6	0	0	0	0	0	0	0	0	0	0	0	2	13	49	140	336	
7	0	0	0	0	0	0	0	0	0	0	0	0	0	2	15	64	
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2

Table 7: The number $c_{n,k}^{(1)}$ of independent k -subsets of $\mathbf{C}_n^{(1)}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
k=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2	0	0	0	0	0	0	3	7	12	18	25	33	42	52	63	75	88	102
3	0	0	0	0	0	0	0	0	0	3	10	22	40	65	98	140	192	255
4	0	0	0	0	0	0	0	0	0	0	0	3	13	35	75	140	238	
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	16	51	

Table 8: The number $c_{n,k}^{(2)}$ of independent k -subsets of $\mathbf{C}_n^{(2)}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
k=0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	0	0	0	0	0	0	0	4	9	15	22	30	39	49	60	72	85	99	
3	0	0	0	0	0	0	0	0	0	0	0	4	13	28	50	80	119	168	
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	17	45	

Table 9: The number $c_{n,k}^{(3)}$ of independent k -subsets of $\mathbf{C}_n^{(3)}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384	32768	65536
1	1	2	3	4	7	11	18	29	47	76	123	199	322	521	843	1364	2207
2	1	2	3	4	5	6	10	15	21	31	46	67	98	144	211	309	453
3	1	2	3	4	5	6	7	8	13	19	26	34	47	66	92	126	173
4	1	2	3	4	5	6	7	8	9	10	16	23	31	40	50	66	89
5	1	2	3	4	5	6	7	8	9	10	11	12	19	27	36	46	57
6	1	2	3	4	5	6	7	8	9	10	11	12	13	14	22	31	41
7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	25
8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
9	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
10	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Table 10: The number $c_n^{(h)}$ of all independent sets of $\mathbf{C}_n^{(h)}$

	n=1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16384
1	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843
2	3	1	1	4	5	6	10	15	21	31	46	67	98	144	211
3	4	1	1	1	5	6	7	8	13	19	26	34	47	66	92
4	5	1	1	1	1	6	7	8	9	10	16	23	31	40	50
5	6	1	1	1	1	1	7	8	9	10	11	12	19	27	36
6	7	1	1	1	1	1	1	8	9	10	11	12	13	14	22
7	8	1	1	1	1	1	1	1	9	10	11	12	13	14	15
8	9	1	1	1	1	1	1	1	1	10	11	12	13	14	15
9	10	1	1	1	1	1	1	1	1	1	11	12	13	14	15
10	11	1	1	1	1	1	1	1	1	1	1	12	13	14	15

Table 11: Values of the h -Lucas sequence $\mathcal{L}^{(h)} = \{L_n^{(h)}\}_{n \geq 1}$

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h=0	0	1	4	12	32	80	192	448	1024	2304	5120	11264	24576	53248	114688	245760
1	0	0	2	3	8	15	30	56	104	189	340	605	1068	1872	3262	5655
2	0	0	0	3	4	5	12	21	32	54	90	143	228	364	574	900
3	0	0	0	0	4	5	6	7	16	27	40	55	84	130	196	285
4	0	0	0	0	0	5	6	7	8	9	20	33	48	65	84	120
5	0	0	0	0	0	0	6	7	8	9	10	11	24	39	56	75
6	0	0	0	0	0	0	0	7	8	9	10	11	12	13	28	45
7	0	0	0	0	0	0	0	0	8	9	10	11	12	13	14	15
8	0	0	0	0	0	0	0	0	0	9	10	11	12	13	14	15
9	0	0	0	0	0	0	0	0	0	0	10	11	12	13	14	15
10	0	0	0	0	0	0	0	0	0	0	0	11	12	13	14	15

Table 12: The number $M_n^{(h)}$ of edges of $\mathbf{M}_n^{(h)}$, for $n > h$

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