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Investigating indipendent subsets of graphs, with Mathematica

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Prologue

This presentation follows step by step the research carried out in

[CD12a] Pietro Codara, Ottavio M. D’Antona, *On the independent subsets of powers of paths and cycles*, arXiv:1210.5561 [cs.DM] (2012)

[CD12b] Pietro Codara, Ottavio M. D’Antona, *Independent subsets of powers of paths, and Fibonacci cubes*, Electronic Notes in Discrete Mathematics (2013), pp. 65-69, DOI 10.1016/j.endm.2013.05.013

...the focus, however, is different.

Prologue

[CD12a] and [CD12b] provide results supported by analytical proofs, as in a classical mathematical paper.

Here we want to show in what stages of our research work, and how, *Mathematica* can help us.

* Enumeration: a *Mathematica* implementation of the formulas allows to observe tabulated results.

* Intuition: a graphical representation of the structures involved allows to have a better intuition about their properties.

* Conjecture: when we are able to conjecture two or more different formulas to perform the same calculation, *Mathematica* can give us immediate feedback about the quality of our hypothesis.

Basic notions

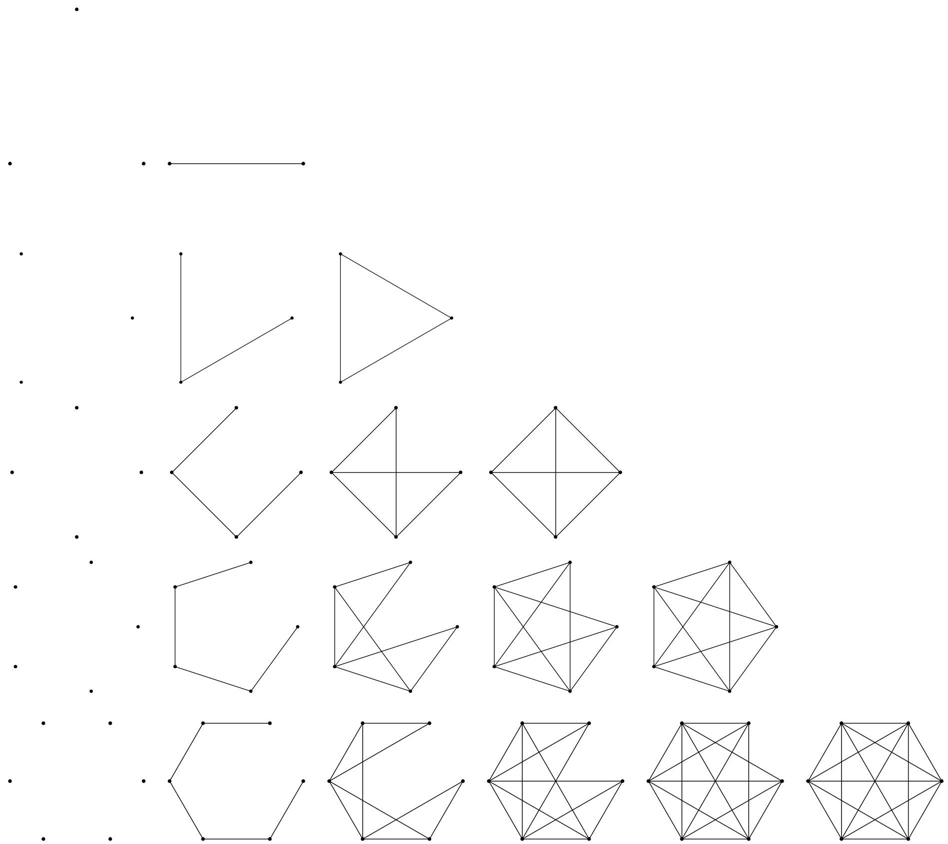
For a graph G we denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its edges.

Definition 1.

For $n, h \geq 0$,
* the *h-power of a path*, denoted by $P_n^{(h)}$, is a graph with n vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n, i \neq j, \left(v_i, v_j\right) \in E\left(P_n^{(h)}\right)$ if and only if $|j-i| \leq h$;
* the *h-power of a cycle*, denoted by $Q_n^{(h)}$, is a graph with n vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n, i \neq j, \left(v_i, v_j\right) \in E\left(Q_n^{(h)}\right)$ if and only if $|j-i| \leq h$ or $|j-i| \geq n-h$.

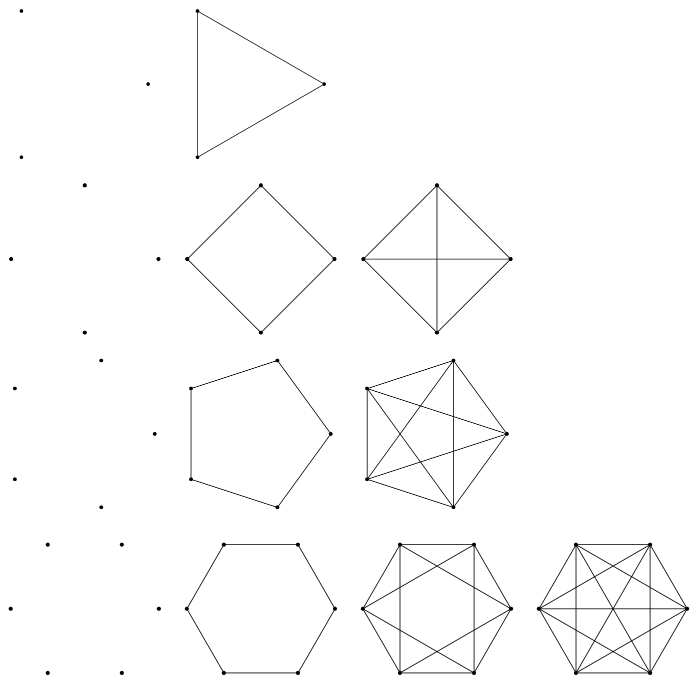
Definition 2.
An *independent subset of a graph* G is a subset of $V(G)$ not containing adjacent vertices.

The h-power of a path



* $P_n^{(0)}$ is a graph made of n isolated nodes
* $P_n^{(1)}$ is the path with n vertices

The h-power of a cycle



* $Q_n^{(0)}$ is a graph made of n isolated nodes
* $Q_n^{(1)}$ is the cycle with n vertices

The independent subsets of powers of paths

We denote by $p_{n,k}^{(h)}$ the number of independent k-subsets of $P_n^{(h)}$.

* For n, h, $k \geq 0$, $p_{n,k}^{(h)} = \binom{n - hk + h}{k}$.

We denote by $p_n^{(h)}$ the number of independent subsets of $P_n^{(h)}$.

* $p_n^{(h)} = \sum_{k \geq 0} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n - hk + h}{k}$

* $p_n^{(h)} = \begin{cases} n+1 & \text{if } n \leq h+1\,, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h+1\,. \end{cases}$

The independent subsets of powers of paths

```
p[h_, n_] := Sum[p[h, n, k], {k, 0, Ceiling[n/(h + 1)]}]
```

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
2	1	2	3	4	6	9	13	19	28	41	60	88	129	189
3	1	2	3	4	5	7	10	14	19	26	36	50	69	95
4	1	2	3	4	5	6	8	11	15	20	26	34	45	60
5	1	2	3	4	5	6	7	9	12	16	21	27	34	43
6	1	2	3	4	5	6	7	8	10	13	17	22	28	35
7	1	2	3	4	5	6	7	8	9	11	14	18	23	29
8	1	2	3	4	5	6	7	8	9	10	12	15	19	24
9	1	2	3	4	5	6	7	8	9	10	11	13	16	20
10	1	2	3	4	5	6	7	8	9	10	11	12	14	17

```
p2[h_, n_] := If[n > h + 1, p[h, n - 1] + p[h, n - h - 1], n + 1]
```

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
2	1	2	3	4	6	9	13	19	28	41	60	88	129	189
3	1	2	3	4	5	7	10	14	19	26	36	50	69	95
4	1	2	3	4	5	6	8	11	15	20	26	34	45	60
5	1	2	3	4	5	6	7	9	12	16	21	27	34	43
6	1	2	3	4	5	6	7	8	10	13	17	22	28	35
7	1	2	3	4	5	6	7	8	9	11	14	18	23	29
8	1	2	3	4	5	6	7	8	9	10	12	15	19	24
9	1	2	3	4	5	6	7	8	9	10	11	13	16	20
10	1	2	3	4	5	6	7	8	9	10	11	12	14	17

Remark

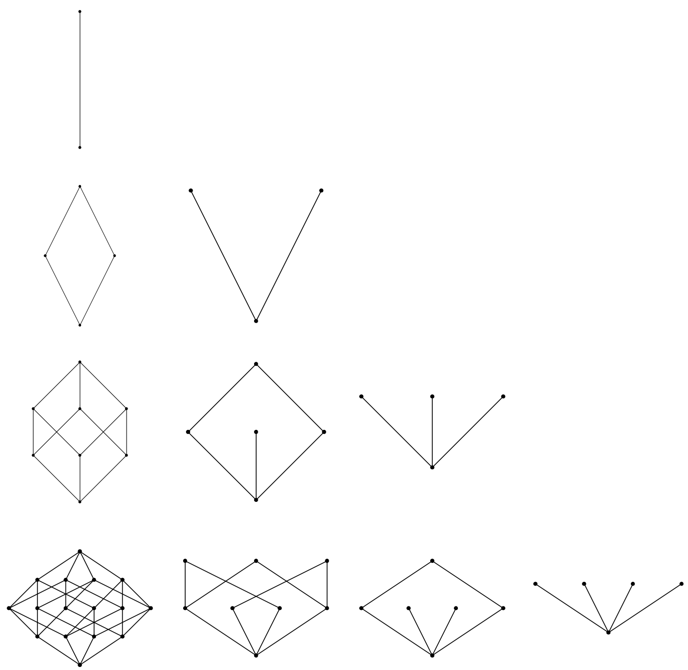
A *Fibonacci string of order n* is a binary strings of length n without (two) consecutive 1's. Recalling that the Hamming distance between two binary strings α and β is the number $H(\alpha, \beta)$ of bits where α and β differ, we can define the *Fibonacci cube of order n*, denoted Γ_n, as the graph (V, E), where V is the set of all Fibonacci strings of order n and, for all $\alpha, \beta \in V$, $(\alpha, \beta) \in E$ if and only if $H(\alpha, \beta) = 1$.

Denote by F_n the n^{th} element of the Fibonacci sequence $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$ (the first values of the Fibonacci sequence are shown below). Then, $p_n^{(1)} = F_{n+2}$ is the number of elements of the Fibonacci cube of order n.

n=1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	2	3	5	8	13	21	34	55	89	144	233

The poset of independent subsets of powers of paths

Let $H_n^{(h)}$ be the Hasse diagram of the poset of independent subsets of $P_n^{(h)}$ ordered by inclusion. The following picture shows $H_n^{(h)}$ for $n = 1, \ldots, 4$, and $h \geq 0$. (Whenever $h \geq n$, $H_n^{(h)}$ coincide with $H_{h-1}^{(h)}$, and it is not shown).

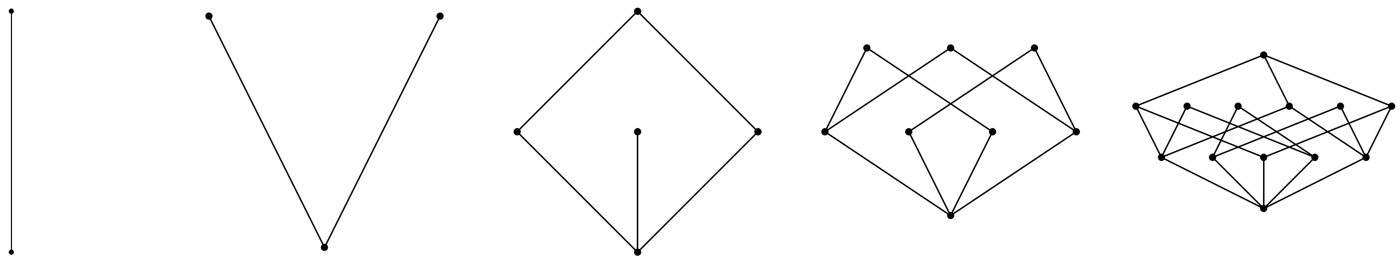


The poset of independent subsets of powers of paths

The first column of the previous picture - $H_n^{(0)}$ - is formed by the n-cubes, and the second column - $H_n^{(1)}$ - is the column of the Fibonacci cubes Γ_n. To check the latter, Fibonacci cubes can also be drawn by mean of their definition in terms of Fibonacci strings.

```
FibonacciCube[n_] := HasseDiagram[
   MakeGraph[FibonacciStrings[n], HammingDistance[#1, #2] == 1 && Count[#1, 1] < Count[#2, 1] &]];

ShowGraphArray[Table[FibonacciCube[n], {n, 1, 5}]]
```



Two ways of counting the edges of $H_n^{(h)}$

Let $H_n^{(h)}$ be the number of edges of $H_n^{(h)}$. Noting that in $H_n^{(h)}$ each non-empty independent k-subset covers exactly k independent $(k-1)$-subsets, we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \, p_{n,k}^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \binom{n - h\,k + h}{k}.$$

Let now $T_{k,i}^{(n,h)}$ be the number of independent k-subsets of $P_n^{(h)}$ containing the vertex v_i. For positive n,

$$\sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = H_n^{(h)} .$$

A generalized Fibonacci sequence

For $h \geq 0$, and $n \geq 1$, we define the *h-Fibonacci sequence* $F^{(h)} = \{F_n^{(h)}\}_{n \geq 1}$ whose elements are

$$F_n^{(h)} = \begin{cases} 1 & \text{if } n \leq h+1\,, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h+1\,. \end{cases}$$

```
F[h_, n_] := If[n ≤ h + 1, 1, F[h, n - 1] + F[h, n - h - 1]]
```

	n=1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16 384
1	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
2	1	1	1	2	3	4	6	9	13	19	28	41	60	88	129
3	1	1	1	1	2	3	4	5	7	10	14	19	26	36	50
4	1	1	1	1	1	2	3	4	5	6	8	11	15	20	26
5	1	1	1	1	1	1	2	3	4	5	6	7	9	12	16
6	1	1	1	1	1	1	1	2	3	4	5	6	7	8	10
7	1	1	1	1	1	1	1	1	2	3	4	5	6	7	8
8	1	1	1	1	1	1	1	1	1	2	3	4	5	6	7
9	1	1	1	1	1	1	1	1	1	1	2	3	4	5	6
10	1	1	1	1	1	1	1	1	1	1	1	2	3	4	5

- $F^{(0)} = 1, 2, 4, \ldots, 2^n, \ldots;$
- $F^{(1)}$ is the Fibonacci sequence;
- more generally, $F^{(h)} = \underbrace{1, \ldots, 1}_{h}, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots.$

One more way of counting the edges of $H_n^{(h)}$

Theorem 1.

For n, $h \geq 0$, we have

$$H_n^{(h)} = \left(F^{(h)} * F^{(h)}\right)(n) = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)}.$$

Remark.

For $h = 1$, we obtain the number of edges of Γ_n by using Fibonacci numbers:

$$H_n^{(1)} = \sum_{i=1}^{n} F_i \, F_{n-i+1} \,.$$

Counting the edges of $H_n^{(h)}$, with Mathematica

```
H[h_, n_] := Sum[k p[h, n, k], {k, 1, Ceiling[n/(h + 1)]}]

pbar[h_, n_, k_] := If[n < 0, p[h, 0, k], p[h, n, k]]

T[h_, n_, k_, i_] := Sum[pbar[h, i - h - 1, r] pbar[h, n - i - h, k - 1 - r] , {r, 0, k - 1}]

H2[h_, n_] := Sum[T[h, n, k, i], {k, 1, Ceiling[n/(h + 1)]}, {i, 1, n}]

F[h_, n_] := If[n ≤ h + 1, 1, F[h, n - 1] + F[h, n - h - 1]]

H3[h_, n_] := Sum[F[h, i] F[h, n - i + 1], {i, 1, n}]
```

...H, H2, H3 lead to the same table.

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13
h=0	0	1	4	12	32	80	192	448	1024	2304	5120	11 264	24 576	53 248
1	0	1	2	5	10	20	38	71	130	235	420	744	1308	2285
2	0	1	2	3	6	11	18	30	50	81	130	208	330	520
3	0	1	2	3	4	7	12	19	28	42	64	97	144	212
4	0	1	2	3	4	5	8	13	20	29	40	56	80	115
5	0	1	2	3	4	5	6	9	14	21	30	41	54	72
6	0	1	2	3	4	5	6	7	10	15	22	31	42	55
7	0	1	2	3	4	5	6	7	8	11	16	23	32	43
8	0	1	2	3	4	5	6	7	8	9	12	17	24	33
9	0	1	2	3	4	5	6	7	8	9	10	13	18	25
10	0	1	2	3	4	5	6	7	8	9	10	11	14	19

The independent subsets of powers of cycles

For n, h, $k \geq 0$, we denote by $q_{n,k}^{(h)}$ the number of independent k-subsets of $Q_n^{(h)}$. The number of all independent subsets of $Q_n^{(h)}$ is

$$q_n^{(h)} = \sum_{k\geq 0} q_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1)\rceil} q_{n,k}^{(h)}\,,$$

$$q_n^{(h)} = \begin{cases} n+1 & \text{if } n \leq 2\,h+1\,, \\ q_{n-1}^{(h)} + q_{n-h-1}^{(h)} & \text{if } n > 2\,h+1. \end{cases}$$

```
q[h_, n_] := Sum[q[h, n, k], {k, 0, Ceiling[n/(h + 1)]}]

q2[h_, n_] := If[n > 2 h + 1, q[h, n - 1] + q[h, n - h - 1], n + 1]
```

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16 384	32 768	65 536	131 072	262 144
1	1	2	3	4	7	11	18	29	47	76	123	199	322	521	843	1364	2207	3571	5778
2	1	2	3	4	5	6	10	15	21	31	46	67	98	144	211	309	453	664	973
3	1	2	3	4	5	6	7	8	13	19	26	34	47	66	92	126	173	239	331
4	1	2	3	4	5	6	7	8	9	10	16	23	31	40	50	66	89	120	160
5	1	2	3	4	5	6	7	8	9	10	11	12	19	27	36	46	57	69	88
6	1	2	3	4	5	6	7	8	9	10	11	12	13	14	22	31	41	52	64
7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	25	35	46
8	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	28
9	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
10	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

Remark

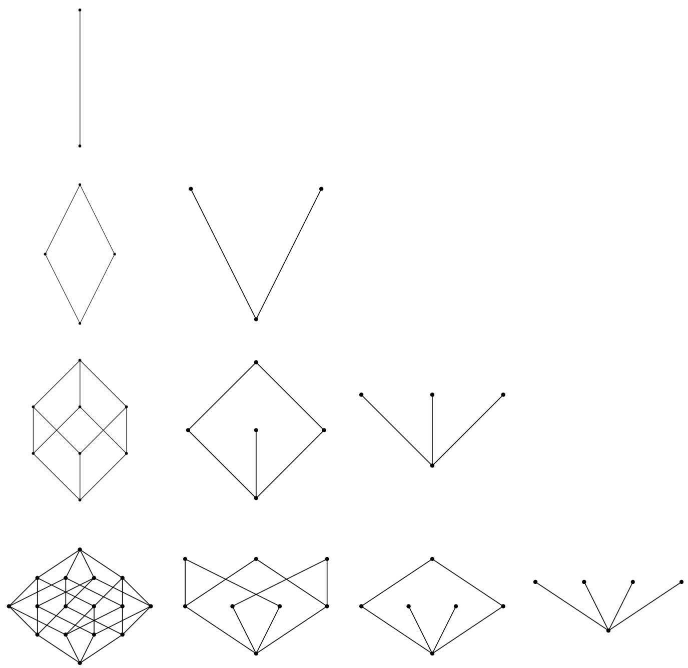
A *Lucas cube of order n*, denoted Λ_n, is defined as the graph whose vertices are the binary strings of length n without either two consecutive 1's or a 1 in the first and in the last position, and in which the vertices are adjacent when their Hamming distance is exactly 1.

Denote by L_n the n^{th} element of the Lucas sequence $L_1 = 1$, $L_2 = 3$, and $L_i = L_{i-1} + L_{i-2}$, for $i > 2$ (some values of the Lucas sequence are shown below). Then, for $n > 1$,
$q_n^{(1)} = L_n$ is the number of elements of the Lucas cube of order n.

n=1	2	3	4	5	6	7	8	9	10	11	12	13
1	3	4	7	11	18	29	47	76	123	199	322	521

The poset of independent subsets of powers of cycles

Let $M_n^{(h)}$ be the Hasse diagram of the poset of independent subsets of $Q_n^{(h)}$ ordered by inclusion. The following picture shows a few Hasse diagrams $M_n^{(h)}$.



* For each n, $M_n^{(1)}$ is the Lucas cube Λ_n.

Toward an analogous of Theorem 1

In order to obtain an analogous of the Theorem 1 for the case of cycles, the first step is to generalize Lucas numbers in an appropriate way. After several attempts, we can assume that the generalization of the Lucas numbers which agrees with our combinatorial structures is the following.

```
L[h_, n_] := If[n ≤ h + 1, 1, L[h, n - 1] + L[h, n - h - 1]]; L[h_, 1] := h + 1
```

	n=1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h=0	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192	16 384
1	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843
2	3	1	1	4	5	6	10	15	21	31	46	67	98	144	211
3	4	1	1	1	5	6	7	8	13	19	26	34	47	66	92
4	5	1	1	1	1	6	7	8	9	10	16	23	31	40	50
5	6	1	1	1	1	1	7	8	9	10	11	12	19	27	36
6	7	1	1	1	1	1	1	8	9	10	11	12	13	14	22
7	8	1	1	1	1	1	1	1	9	10	11	12	13	14	15
8	9	1	1	1	1	1	1	1	1	10	11	12	13	14	15
9	10	1	1	1	1	1	1	1	1	1	11	12	13	14	15
10	11	1	1	1	1	1	1	1	1	1	1	12	13	14	15

We call this sequences the *h-Lucas sequences*, and denote them by $L^{(h)} = \left\{L_n^{(h)}\right\}_{n\geq 1}$. Notice that $L_{n+1}^{(1)} = L_n$ is the sequence of Lucas numbers, for each $n \geq 1$.

Toward an analogous of Theorem 1

The following function seems to be able to generate, although only for the case $n \geq h$, our values $M_n^{(h)}$, by using an appropriate discrete convolution of an *h*-Lucas sequence and an *h*-Fibonacci sequence. The latter fact can be checked by comparing the following table with the table of values of $M_n^{(h)}$ provided by the formula $M_n^{(h)} = n \sum_{k=0}^{\lceil n/h+1 \rceil} \binom{n-h\,k-1}{k-1}$.

```
Mtent[h_, n_] := Sum[F[h, i] L[h, n - h - i + 1], {i, 1, n - h}]
```

	n=0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
h=0	0	1	4	12	32	80	192	448	1024	2304	5120	11 264	24 576	53 248	114 688	245 760
1	0	0	2	3	8	15	30	56	104	189	340	605	1068	1872	3262	5655
2	0	0	0	3	4	5	12	21	32	54	90	143	228	364	574	900
3	0	0	0	0	4	5	6	7	16	27	40	55	84	130	196	285
4	0	0	0	0	0	5	6	7	8	9	20	33	48	65	84	120
5	0	0	0	0	0	0	6	7	8	9	10	11	24	39	56	75
6	0	0	0	0	0	0	0	7	8	9	10	11	12	13	28	45
7	0	0	0	0	0	0	0	0	8	9	10	11	12	13	14	15
8	0	0	0	0	0	0	0	0	0	9	10	11	12	13	14	15
9	0	0	0	0	0	0	0	0	0	0	10	11	12	13	14	15
10	0	0	0	0	0	0	0	0	0	0	0	11	12	13	14	15

Conjecture.

For $n > h \geq 0$, we have

$$M_n^{(h)} = \left(F^{(h)} * L^{(h)}\right)(n-h)\,.$$

Epilogue

Mathematica turns out to be an useful tool for research in combinatorics, especially in enumerative combinatorics.

Indeed, with *Mathematica* we can tabulate results, check formulas, get a clearer intuition on the combinatorial structures involved... and even conjecture.

Note that we have used *Mathematica* just as a tool. Indeed, *Mathematica* can help, as shown, in our theoretical studies, but the final product of our work is not a *Mathematica* notebook, but rather a scientific paper, with results supported by analytical proofs.

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[CD12b] Pietro Codara, Ottavio M. D'Antona, *Independent subsets of powers of paths, and Fibonacci cubes*, Electronic Notes in Discrete Mathematics, Vol. 40, 2013, pp. 65-69, ISSN 1571-0653, 10.1016/j.endm.2013.05.013

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