Mathematica Italia
User Group Meeting

A *Mathematica* package to cope with partially ordered sets

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Outline

1. Introduction to *poset*

2. Basic features

3. Investigating partitions of posets

4. Investigating the lattice of partitions

5. Other features

The package *poset*

- basic features to treat *partially ordered sets*
- enumerate, create, and display *monotone and regular partitions* of partially ordered sets
- deal with the *lattices* of partitions of a poset
- compute *products* and *coproducts* in the category *of partially ordered sets* and monotone maps
- compute *products* and *coproducts* in the category *of forests* (disjoint union of trees) and open maps

Basic features (1)

```
<< poset.m
```

```
? Poset
```

Poset[*relation*] and Poset[*relation*],*v*] generate a partially ordered set, represented as a directed graph (see *Combinatorica* manual). *relation* is a set of pairs representing the order relation of the poset (not necessarily the entire order relation). *v* is a list of vertices.

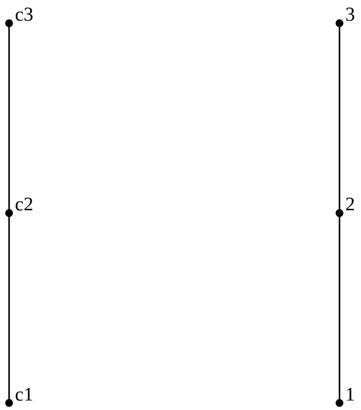
```
C3 = Poset[{{c1, c2}, {c2, c3}}]
```

```
- Graph:< 6,3,Directed >-
```

```
CU3 = Chain[3]
```

```
- Graph:< 6,3,Directed >-
```

```
Hasse[{C3, CU3}]
```

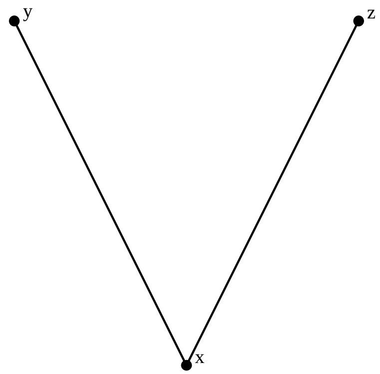


Basic features (2)

```
B2 = Poset[{{"x", "y"}, {"x", "z"}}]
```

- Graph:< 5,3,Directed >-

```
Hasse[B2]
```




```
Relation[B2]
```

{{x, x}, {x, y}, {x, z}, {y, y}, {z, z}}

```
Covering[B2]
```

{{x, y}, {x, z}}

```
PosetElements[B2]
```

{x, y, z}

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Investigating partitions of poset

[Cod08] Pietro Codara, *A theory of partitions of partially ordered sets*, Ph.D. thesis, Università degli Studi di Milano, Italy (2008).

[Cod09] Pietro Codara, *Partitions of a finite partially ordered set*, From Combinatorics to Philosophy: The Legacy of G.-C. Rota, Springer, New York (2009), 45--59.

```
? PosetPartitions
```

PosetPartitions[*p*] generates the list of all monotone partitions of a poset. Each monotone partition is represented as a *graph*. *p* is a *poset*. PosetPartitions[*p*] outputs a list of *graphs*, and displays the total number of monotone partition of *p*, and the total number of cases analyzed by the function to obtain the monotone partitions.

```
? RegularPartitions
```

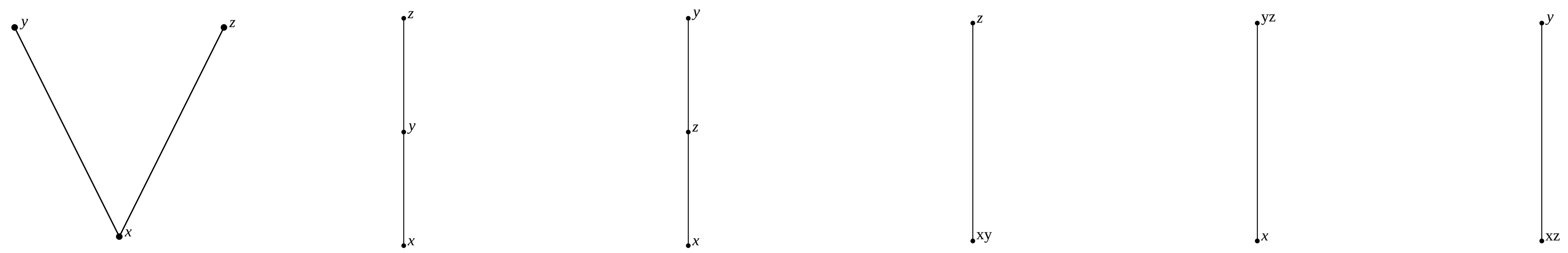
RegularPartitions[*p*] generates the list of all regular partitions of a poset. Each regular partition is represented as a *graph*. *p* is a *poset*. RegularPartitions[*p*] outputs a list of *graphs*, and displays the total number of regular partition of *p*, and the total number of cases analyzed by the function to obtain the regular partitions.

Monotone and regular partitions of a poset

```
PB2 = PosetPartitions[B2];
```

Analyzed preorders: 16 – Poset Partitions: 7

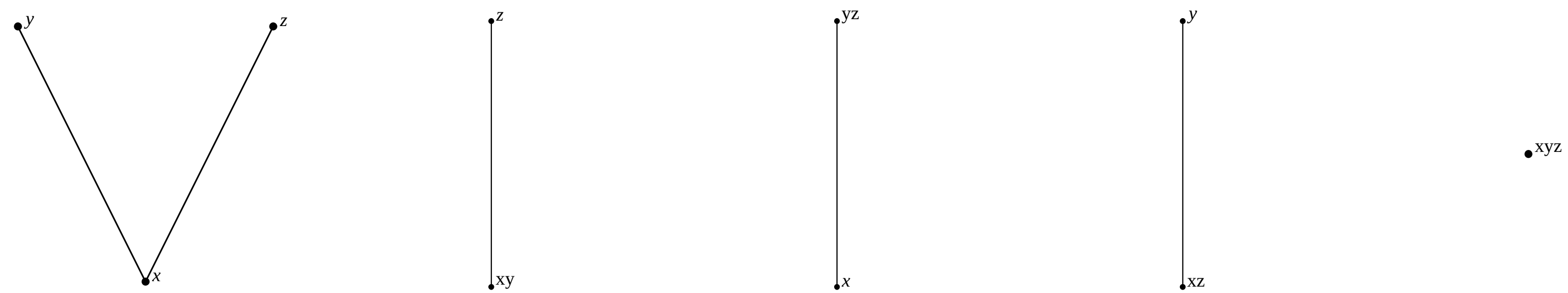
```
CreateHasse[PB2, 8]
```




```
RB2 = RegularPartitions[B2];
```

Analyzed preorders: 5 – Regular Partitions: 5

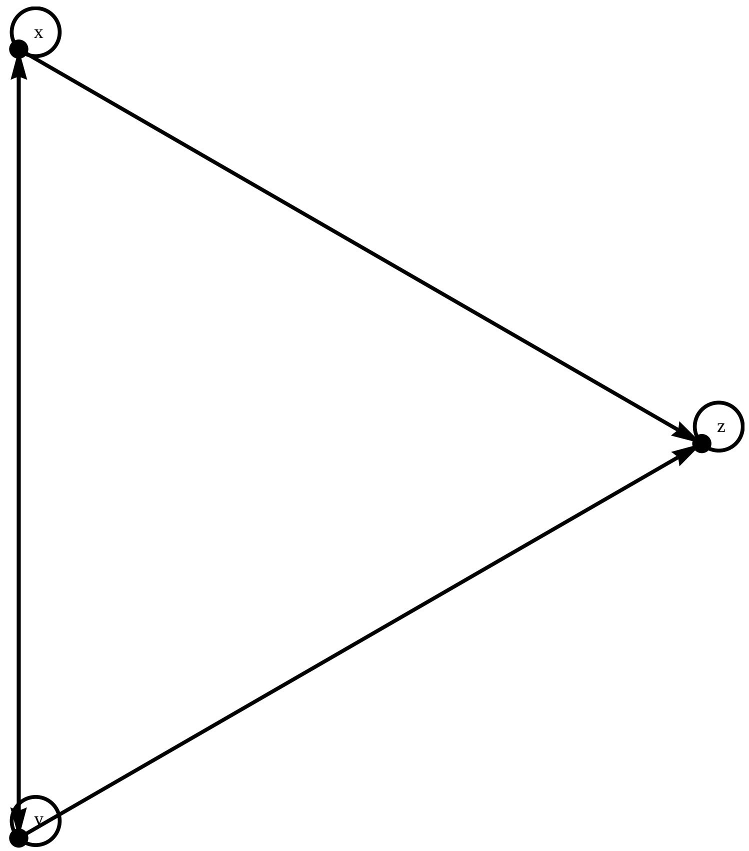
```
CreateHasse[RB2, 6]
```



Partitions of a poset: note on implemetation (1)

- In general, a monotone partitition returned by the function *PosetPartitions* is not a poset, but a preorder (i.e. the binary relation does not have the antisymmetric property).
- If we apply the function *Hasse* to one of such partitions, we do not obtain an Hasse diagram, but a directed graph.

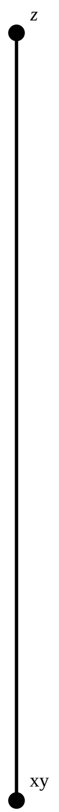
```
Hasse[PB2[[4]]]
```



Partitions of a poset: note on implemetation (2)

- The function *PartitionToPoset* solves this problem, by reducing blocks to single elements and concatenating labels.

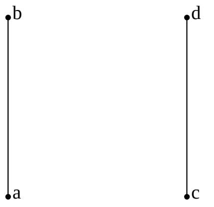
```
Hasse[PartitionToPoset[PB2[[4]]]]
```



Regular partitions

- The poset...

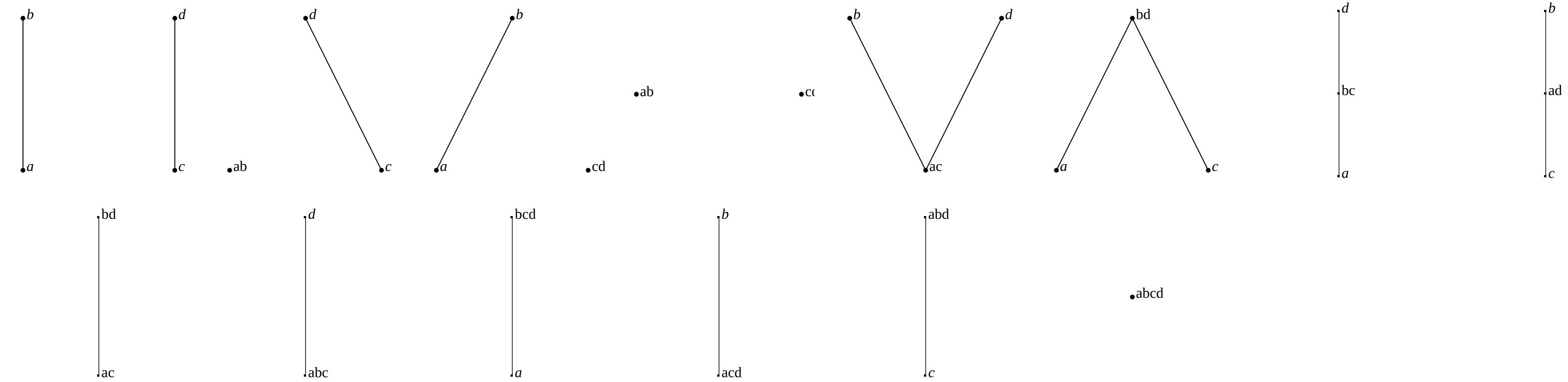
```
Hasse[P = Poset[{{"a", "b"}, {"c", "d"}}]]
```



- The regular partitions...

```
CreateHasse[RegularPartitions[P], 8]
```

```
Analyzed preorders: 15 – Regular Partitions: 14
```

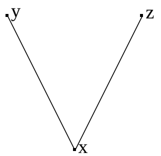


The lattice of regular partitions

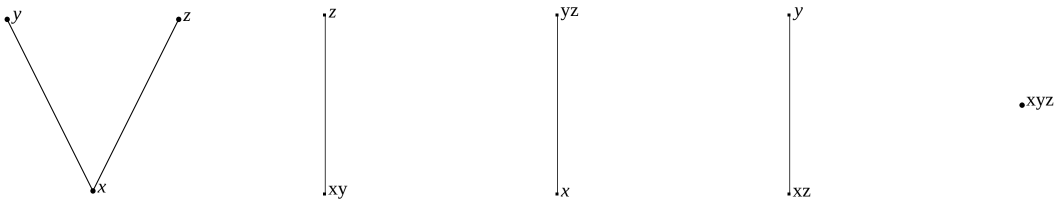
```
? PosetPartitionLattice
```

PosetPartitionLattice[*plist*] returns the lattice structure of a given list of monotone or regular partitions *plist*. *plist* is usually obtained by using *RegularPartitions* or *PosetPartitions*.

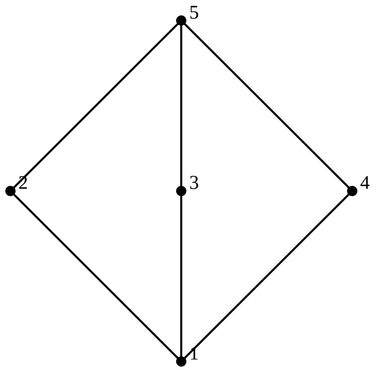
- The poset...



- The regular partitions...

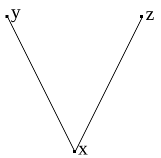



```
CreateHasse[MP = PosetPartitionLattice[RB2]]
```

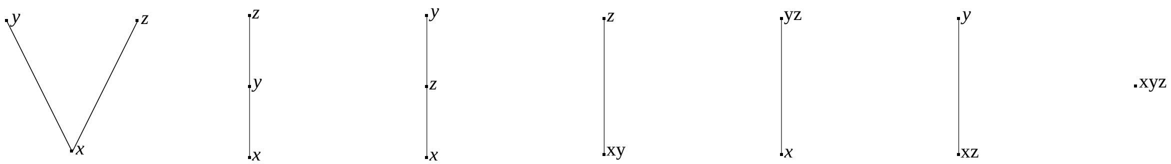


The lattice of monotone partitions

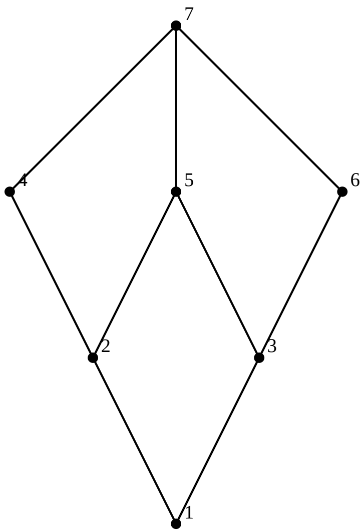
- The poset...



- The monotone partitions...



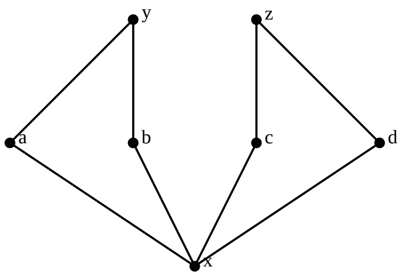

```
CreateHasse[RP = PosetPartitionLattice[PB2]]
```



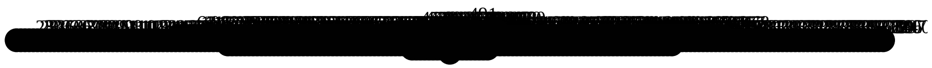
Additional features

Some functions of the package are particularly useful when the graphical output cannot offer any information on the lattice.

- The poset...



- The Lattice of regular partitions...



Investigating the properties of the lattice

```
BigLattice = RegularPartitions[P4];
```

Analyzed preorders: 877 – Regular Partitions: 491

```
WhitneyNumbers[BigLattice]
```

{1, 19, 107, 208, 131, 24, 1}

```
Atoms = AtomsPosition[BigLattice]
```

{2, 3, 4, 5, 6, 7, 15, 16, 20, 21, 23, 24, 25, 26, 33, 49, 50, 51, 61}

```
Coatoms = CoatomsPosition[BigLattice]
```

{457, 458, 459, 460, 461, 462, 463, 464, 465, 466, 467, 468, 479, 480, 481, 482, 483, 484, 485, 486, 487, 488, 489, 490}

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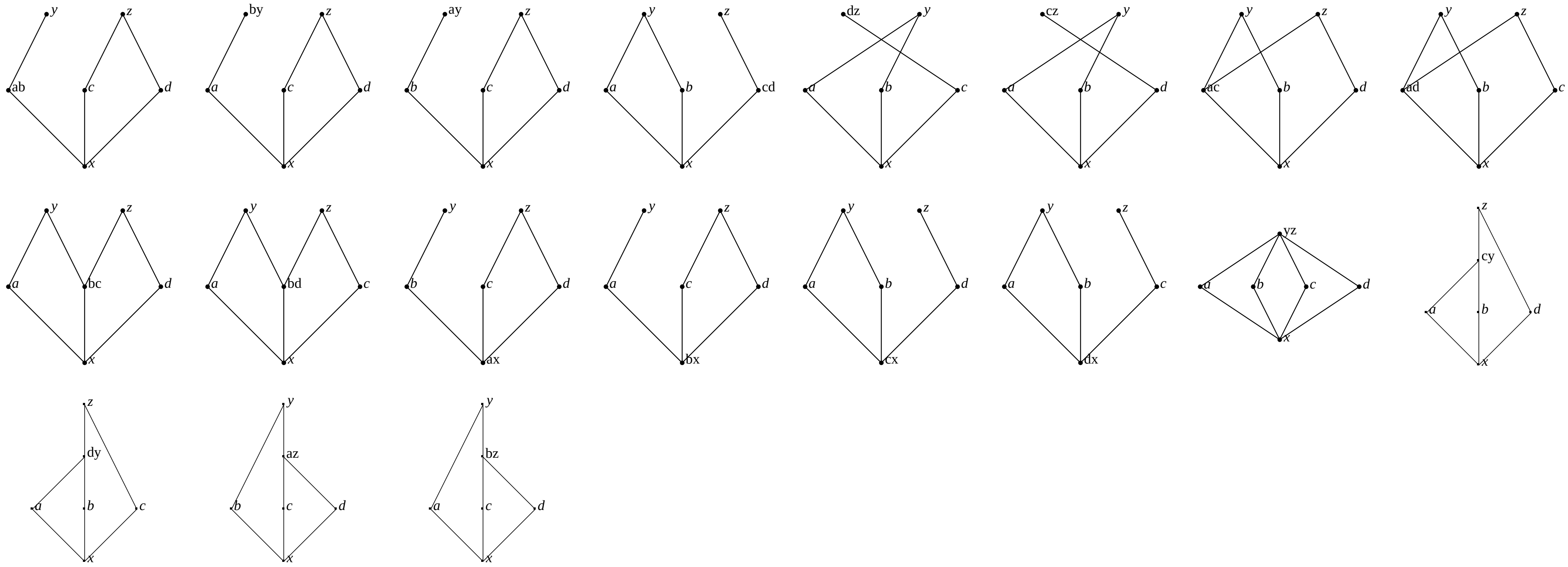
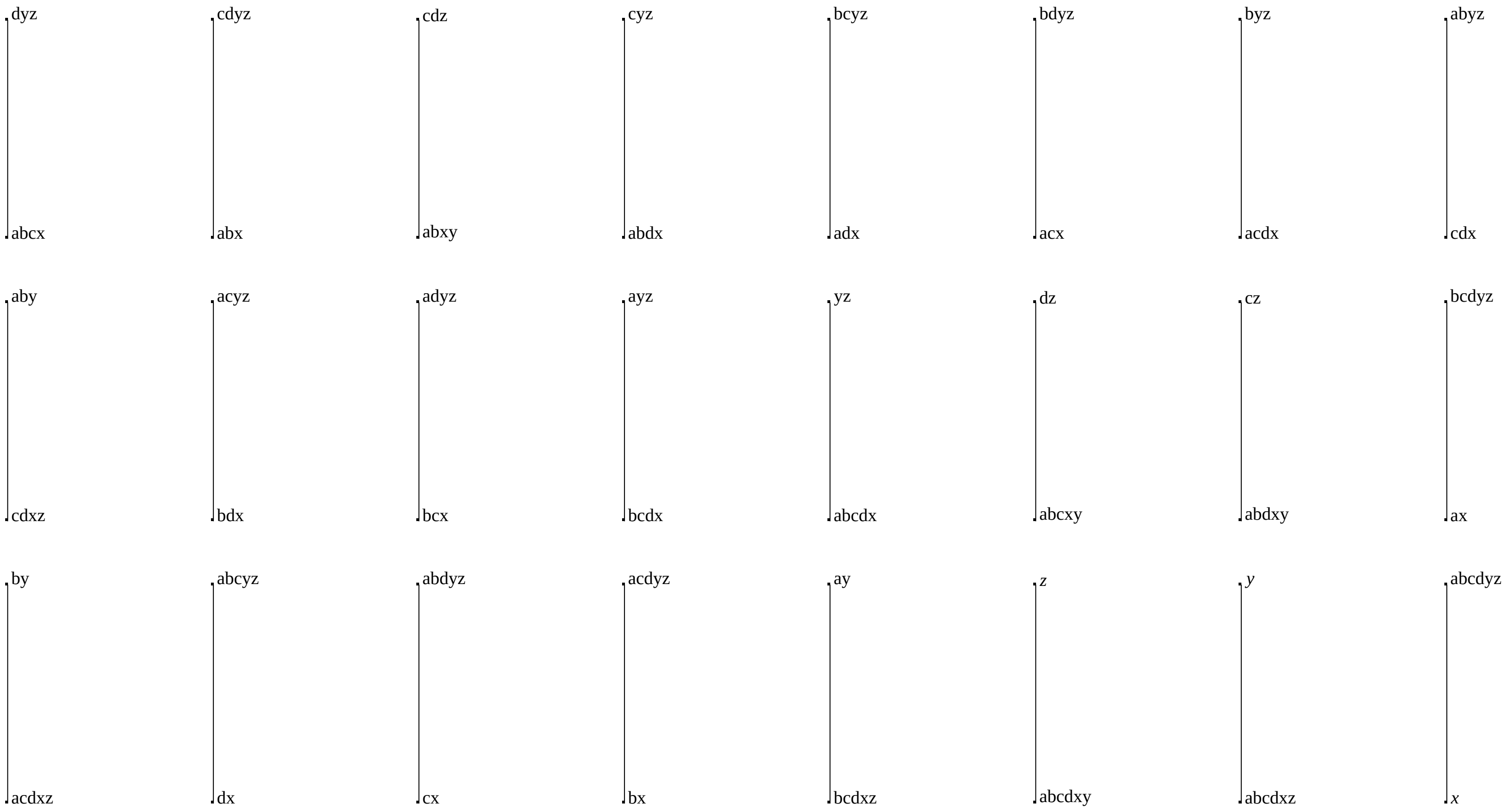
Atoms

```
CreateHasse[BigLattice[[Atoms]], 8]
```



Coatoms

```
CreateHasse[BigLattice[[Coatoms]], 8]
```



Partitions of chains (1)

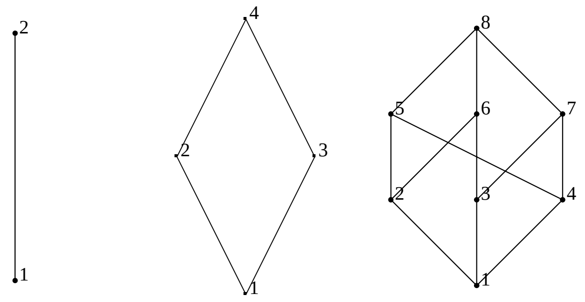
In [Cod08, 6.2] it is proved that the monotone partition lattice and the regular partition lattice of a chain with n elements are isomorphic, and that they are isomorphic to the Boolean lattice B_{n-1}.

```
Hasse[{PosetPartitionLattice[PosetPartitions[Chain[2]]], PosetPartitionLattice[PosetPartitions[Chain[3]]], PosetPartitionLattice[PosetPartitions[Chain[4]]]}]
```

```
Analyzed preorders: 2 – Poset Partitions: 2

Analyzed preorders: 8 – Poset Partitions: 4

Analyzed preorders: 64 – Poset Partitions: 8
```



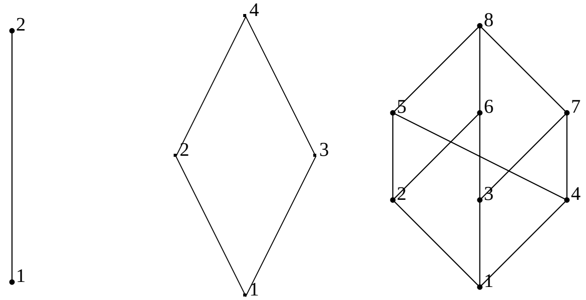
Partitions of chains (2)

```
Hasse[{PosetPartitionLattice[RegularPartitions[Chain[2]]],
  PosetPartitionLattice[RegularPartitions[Chain[3]]], PosetPartitionLattice[RegularPartitions[Chain[4]]]}]
```

Analyzed preorders: 2 – Regular Partitions: 2

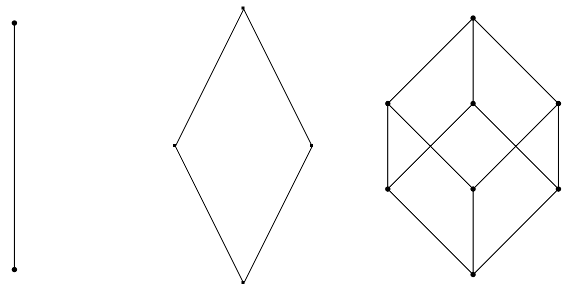
Analyzed preorders: 5 – Regular Partitions: 4

Analyzed preorders: 15 – Regular Partitions: 8



Partitions of chains (3)

```
Hasse[{BooleanAlgebra[1], BooleanAlgebra[2], BooleanAlgebra[3]}]
```

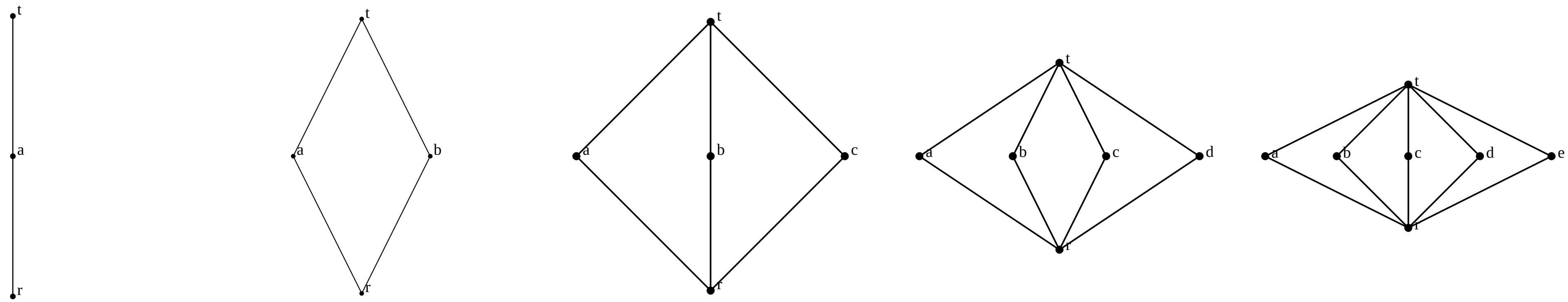


A case of counting (1)

We study an enumerating problem solved in [Cod08, 6.4]. We want to count all regular partitions of a family of posets M_1, M_2, M_3, ...

```
M1 = Poset[{{"r", "a"}, {"a", "t"}}]; M2 = Poset[{{"r", "a"}, {"r", "b"}, {"b", "t"}, {"a", "t"}}];
M3 = Poset[{{"r", "a"}, {"r", "b"}, {"r", "c"}, {"c", "t"}, {"b", "t"}, {"a", "t"}}];
M4 = Poset[{{"r", "a"}, {"r", "b"}, {"r", "c"}, {"r", "d"}, {"d", "t"}, {"c", "t"}, {"b", "t"}, {"a", "t"}}];
M5 = Poset[{{"r", "a"}, {"r", "b"}, {"r", "c"}, {"r", "d"}, {"r", "e"}, {"e", "t"}, {"d", "t"}, {"c", "t"}, {"b", "t"}, {"a", "t"}}];

Hasse[{M1, M2, M3, M4, M5}, 6]
```



A case of counting (2)

```
RegularPartitions[#] & /@ {M1, M2, M3, M4, M5};
```

```
Analyzed preorders: 5 – Regular Partitions: 4

Analyzed preorders: 15 – Regular Partitions: 11

Analyzed preorders: 52 – Regular Partitions: 38

Analyzed preorders: 203 – Regular Partitions: 152

Analyzed preorders: 877 – Regular Partitions: 675
```

The following formula count the total number of regular partitions of the poset M_i .

$$B_{i+2} - B_{i+1} + 1$$

The number B_n is the n^{th} Bell number, and it is computed by the *Mathematica* function *BellB[n]*.

```
Table[BellB[n + 2] - BellB[n + 1] + 1, {n, 1, 15}]
```

```
{4, 11, 38, 152, 675, 3264, 17 008, 94 829, 562 596, 3 535 028, 23 430 841, 163 254 886, 1 192 059 224, 9 097 183 603, 72 384 727 658}
```

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Thank you for your attention

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