Euler Characteristic in Gödel and Nilpotent Minimum Logics

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The Euler-Klee-Rota lattice-theoretic characteristic

Valuation

Let L be a (bounded) distributive lattice whose bottom element is denoted $\perp$. A function $\nu\colon L \to \mathbb{R}$ is a valuation if it satisfies $\nu(\perp) = 0$, and

$$\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y)$$

for all $x, y \in L$.

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for all $x, y \in L$.

Lemma

Every valuation on a finite distributive lattice L is uniquely determined by its values at the join-irreducibles of L.

Recall that $x \in L$ is *join-irreducible* if it is not the bottom of L, and $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$.

The Euler-Klee-Rota lattice-theoretic characteristic, official definition:

(V. Klee 1963; G.-C. Rota 1974)

Euler characteristic

The Euler characteristic of a finite distributive lattice L is the unique valuation $\chi\colon L \to \mathbb{R}$ such that $\chi(x) = 1$ for any join-irreducible element $x \in L$.

The Euler-Klee-Rota lattice-theoretic characteristic

- Let V be a set of *vertices*, and let P be the poset of subsets of V ordered by inclusion. The collection $\mathscr{L}$ of lower sets of P is a (bounded) distributive lattice under $\cap$, $\cup$.

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- An element $\Sigma \in \mathscr{L}$ is the same thing as a (combinatorial) simplicial complex: a collection of subsets of V such that $A \subseteq B \in \Sigma \Rightarrow A \in \Sigma$.

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- An element $\Sigma \in \mathscr{L}$ is the same thing as a (combinatorial) simplicial complex: a collection of subsets of V such that $A \subseteq B \in \Sigma \Rightarrow A \in \Sigma$.
- It turns out that the Euler characteristic of any simplicial complex Σ whose vertices are contained in V can be described in terms of the lattice $\mathscr{L}$.

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- Equivalently, let $\chi\colon \mathscr{L} \to \mathbb{R}$ be such that $\chi(\emptyset) = 0$, and $\chi(\Delta) = 1$ whenever Δ is a simplex.
- It turns our that χ as in the above agrees with the classical Euler characteristic on each simplicial complex $\Sigma \in \mathscr{L}$.

Outline

1. Euler Characteristic of a formula in classical propositional logic
2. Euler Characteristic of a formula in Gödel logic
3. Euler Characteristic of a formula in Nilpotent Minimum logic

Euler characteristic of a classical formula

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- Writing $\equiv$ for the relation of logical equivalence, the quotient set $\textsc{Form}_n/\equiv$ is naturally a Boolean algebra.
- So we can consider valuations on $\textsc{Form}_n/\equiv$. In particular, let χ be the Euler(-Klee-Rota) characteristic of $\textsc{Form}_n/\equiv$.

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- So we can consider valuations on $\textsc{Form}_n/\equiv$. In particular, let χ be the Euler(-Klee-Rota) characteristic of $\textsc{Form}_n/\equiv$.
- Then we say that *the Euler characteristic of* φ *is* $\chi([\varphi]_{\equiv})$.

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$\chi([\varphi]_\equiv)$ *is the number of assignments that satisfy* φ.

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$$\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$$

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They provide the equivalent algebraic semantics of Gödel logic. For an integer $n \geq 0$, let us write $\mathscr{G}_n$ for the Tarski-Lindenbaum algebra of Gödel logic over the variables $X_1, \ldots, X_n$, that is, the algebra $\textsc{Form}_n / \equiv$, where $\equiv$ is the logical equivalence between formulæ.

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Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \textsc{Form}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$.

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In the sense given by this result, the characteristic of a formula as defined above is a classical notion – it will not distinguish, for instance, classical from non-classical tautologies.

Gödel $(k+1)$-valued logic

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$\mathbb{G}_{k+1}$ is obtained from $\mathbb{G}_{\infty}$, Gödel (infinite-valued) logic recalled above, by restricting assignments to those taking values in the set

$$V_{k+1} = \{0 = \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \frac{k}{k} = 1\} \subseteq [0,1] \ ,$$

that is, to $(k+1)$-*valued assignments*.

Generalised Euler characteristic of a formula in Gödel logic

For a join-irreducible $g \in \mathscr{G}_n$, say g has height $h(g)$ if the (unique) chain of join-irreducibles below g in $\mathscr{G}_n$ has cardinality $h(g)$.

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Generalised Euler characteristic

Fix integers $n, k \geq 1$. We write $\chi_k \colon \mathscr{G}_n \to \mathbb{R}$ for the unique valuation on $\mathscr{G}_n$ that satisfies

$$\chi_k(g) = \min\{h(g), k\}$$

for each join-irreducible element $g \in \mathscr{G}_n$. Further, if $\varphi \in \textsc{Form}_n$, we define $\chi_k(\varphi) = \chi_k([\varphi]_\equiv)$.

It turns out that χ_k is a "k-valued characteristic", as we proceed to show.

n-equivalence

Our next aim is to relate χ_k with (not necessarily Boolean) $[0, 1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu\colon \{X_1\} \to [0, 1]$. However, in Gödel logic this fact is quite misleading, and there is the following important reduction to finiteness.

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n-equivalence

Fix integers $n, k \geq 1$. We say that two $(k+1)$-valued assignments μ and ν are *equivalent over the first n variables*, or just *n-equivalent*, if and only if for all formulæ $\varphi(X_1, \ldots, X_n)$ of $\mathbb{G}_{k+1}$, $\mu(\varphi) = 1$ if and only if $\nu(\varphi) = 1$.
The same definition can be given, *mutatis mutandis*, for $\mathbb{G}_\infty$.

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where $P(n,k) = \sum_{i=1}^{k} \sum_{j=0}^{n} \binom{n}{j} T(j,i)$, and

$$T(n,k) = \begin{cases} 1 & \text{if } k = 1\,, \\ 0 & \text{if } k > n+1\,, \\ \displaystyle\sum_{i=1}^{n} \binom{n}{i} T(n-i, k-1) & \text{otherwise}\,. \end{cases}$$

Table

	k=1	2	3	4	5	6	7
n=1	2	3	3	3	3	3	3
2	4	9	11	11	11	11	11
3	8	27	45	51	51	51	51
4	16	81	191	275	299	299	299
5	32	243	813	1563	2043	2163	2163
6	64	729	3431	8891	14771	18011	18731
7	128	2187	14325	49731	106851	158931	184131
8	256	6561	59231	272675	757019	1407179	1921259
9	512	19683	242973	1468203	5228043	12200883	20214483

The number $P(n,k)$ of distinct equivalence classes of $(k+1)$-valued assignments over n variables.

Main result

Theorem

Fix integers $n, k \geq 1$, and a formula $\varphi \in \textsc{Form}_n$.

1. $\chi_k(\varphi)$ equals the number of $(k+1)$-valued assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$, up to n-equivalence.

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2. φ is a tautology in $\mathbb{G}_{k+1}$ if and only if $\chi_k(\varphi) = P(n,k)$.

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2. φ is a tautology in $\mathbb{G}_{k+1}$ if and only if $\chi_k(\varphi) = P(n,k)$.
3. φ is a tautology in $\mathbb{G}_\infty$ if and only if it is a tautology in $\mathbb{G}_{n+2}$ if and only if $\chi_{n+1}(\varphi) = P(n, n+1)$.

Proof of main result

Lemma 1

Fix integers $n, k \geq 1$, let $x \in \mathscr{G}_n$ and consider the valuation $\chi_k \colon \mathscr{G}_n \to \mathbb{R}$. Then, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq x$ and $h(g) \leq k$.

Proof of main result

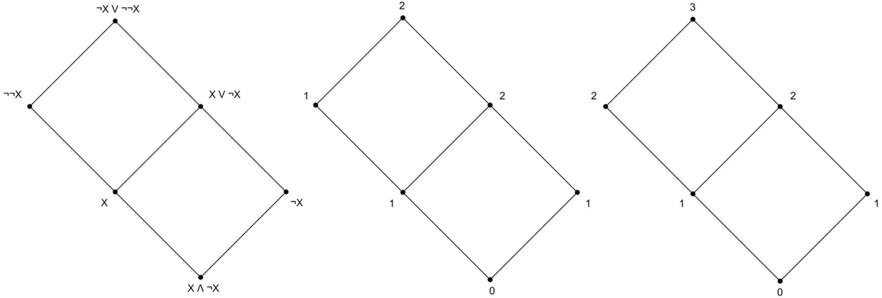
Lemma 1

Fix integers $n, k \geq 1$, let $x \in \mathscr{G}_n$ and consider the valuation $\chi_k : \mathscr{G}_n \to \mathbb{R}$. Then, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq x$ and $h(g) \leq k$.

Lemma 2

Fix integers $n, k \geq 1$, and let $\varphi \in \textsc{Form}_n$. Let $O(\varphi, n, k)$ be the set of equivalence classes $[\mu]_{\equiv^k_n}$ of $(k+1)$-valued assignments such that $\mu(\varphi) = 1$. Further, let $J(\varphi, n, k)$ be the set of join-irreducible elements $x \in \mathscr{G}_n$ such that $x \leq [\varphi]_{\equiv}$ and $h(x) \leq k$. Then there is a bijection between $O(\varphi, n, k)$ and $J(\varphi, n, k)$.

Example



The Gödel algebra $\mathscr{G}_1$, and the values of $\chi = \chi_1 : \mathscr{G}_1 \to \mathbb{R}$ and $\chi_2 : \mathscr{G}_1 \to \mathbb{R}$.

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$$\mu(\alpha \odot \beta) = \begin{cases} \min\{\mu(\alpha), \mu(\beta)\} & \text{if } \mu(\alpha) + \mu(\beta) > 1 \\ 0 & \text{otherwise} \end{cases}$$

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We can now hope that the Euler characteristic of a formula φ can encode logical information similar to that encoded by the characteristic in the case of Gödel logic.
Unfortunately, this is not the case. Indeed, take, for instance, the formula

$$\alpha = (X \leftrightarrow X)^2 \wedge X$$

It turns out that:

- For every assignments $\mu \colon \textsc{Form}_n \to [0,1]$, $\mu(\alpha) < 1$, but
- $[\alpha]_\equiv$ is a join irreducible and thus $\chi(\alpha) = 1$.

Positive Euler characteristic of a formula in NM logic

For an element $x \in \mathscr{NM}_n$, say x is idempotent if $x \odot x = x$.

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Positive Euler characteristic

We write $\chi^+ \colon \mathscr{NM}_n \to \mathbb{R}$ for the unique valuation on $\mathscr{NM}_n$ that satisfies:

1. $\chi^+(x) = 1$ for each idempotent join irreducible element $x \in \mathscr{NM}_n$.
2. $\chi^+(x \odot x) = \chi^+(x)$ for each $x \in \mathscr{NM}_n$;

Further, if $\varphi \in \textsc{Form}_n$, we define $\chi^+(\varphi) = \chi^+([\varphi]_\equiv)$.

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Note that $\alpha \odot \alpha = \bot$, thus $\chi^+(\alpha) = 0$.

Main result

Theorem

Fix $n \geq 1$, and a formula $\varphi \in \textsc{Form}_n$.

$\chi^+(\varphi)$ equals the number of assignments $\mu\colon \textsc{Form}_n \to \{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$.

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Remark. If φ is a tautology in $\mathbb{NM}$, then $\chi^+(\varphi) = 3^n$.

Proof of main result

Lemma 1

Fix $n \geq 1$. Let $x \in \mathscr{NM}_n$ and consider the valuation $\chi^+ \colon \mathscr{NM}_n \to \mathbb{R}$. Then, $\chi^+(x)$ equals the number of minimal idempotent join-irreducible elements $g \in \mathscr{NM}_n$ such that $g \leq x$.

Proof of main result

Lemma 1

Fix $n \geq 1$. Let $x \in \mathcal{NM}_n$ and consider the valuation $\chi^+ \colon \mathcal{NM}_n \to \mathbb{R}$. Then, $\chi^+(x)$ equals the number of minimal idempotent join-irreducible elements $g \in \mathcal{NM}_n$ such that $g \leq x$.

Lemma 2

Fix $n \geq 1$, and let $\varphi \in \textsc{Form}_n$. There is a bijection between the set of equivalence classes $[\mu]_{\equiv_n}$ of assignments to $\{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$ and the set of idempotent join-irreducible elements $x \in \mathcal{NM}_n$ such that $x \leq [\varphi]_{\equiv_n}$.

Further research

- Investigate a generalised positive Euler characteristic for NM logic, as done for Gödel logic.
- Investigate the logical content of the Euler characteristic in NM logic.
- Investigate the Euler characteristic in $\mathbb{NM}^-$.

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Thank you for your attention.