Euler Characteristic in Gödel and Nilpotent Minimum Logics

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The Euler-Klee-Rota lattice-theoretic characteristic

**Valuation**

Let $L$ be a (bounded) distributive lattice whose bottom element is denoted $\bot$. A function $\nu: L \to \mathbb{R}$ is a *valuation* if it satisfies $\nu(\bot) = 0$, and

$$\nu(x) + \nu(y) = \nu(x \lor y) + \nu(x \land y)$$

for all $x, y \in L$. 
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Lemma

Every valuation on a finite distributive lattice $L$ is uniquely determined by its values at the join-irreducibles of $L$.

Recall that $x \in L$ is join-irreducible if it is not the bottom of $L$, and $x = y \lor z$ implies $x = y$ or $x = z$ for all $y, z \in L$. 
The Euler-Klee-Rota lattice-theoretic characteristic, official definition:

(V. Klee 1963; G.-C. Rota 1974)

**Euler characteristic**

The **Euler characteristic** of a finite distributive lattice $L$ is the unique valuation $\chi: L \to \mathbb{R}$ such that $\chi(x) = 1$ for any join-irreducible element $x \in L$. 
The Euler-Klee-Rota lattice-theoretic characteristic

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An element \( \Sigma \in \mathcal{L} \) is the same thing as a (combinatorial) simplicial complex: a collection of subsets of \( V \) such that \( A \subseteq B \in \Sigma \Rightarrow A \in \Sigma \).
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- An element $\Sigma \in \mathcal{L}$ is the same thing as a (combinatorial) simplicial complex: a collection of subsets of $V$ such that $A \subseteq B \in \Sigma \Rightarrow A \in \Sigma$.

- It turns out that the Euler characteristic of any simplicial complex $\Sigma$ whose vertices are contained in $V$ can be described in terms of the lattice $\mathcal{L}$.
Consider the Euler Characteristic on $L$, that is, the unique valuation such that $\chi(\Delta) = 1$ whenever $\Delta$ is a join-irreducible element of the lattice $L$. 
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- Equivalently, let $\chi: \mathcal{L} \rightarrow \mathbb{R}$ be such that $\chi(\emptyset) = 0$, and $\chi(\Delta) = 1$ whenever $\Delta$ is a simplex.
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Equivalently, let $\chi: \mathcal{L} \rightarrow \mathbb{R}$ be such that $\chi(\emptyset) = 0$, and $\chi(\Delta) = 1$ whenever $\Delta$ is a simplex.

It turns our that $\chi$ as in the above agrees with the classical Euler characteristic on each simplicial complex $\Sigma \in \mathcal{L}$. 
Outline

1. Euler Characteristic of a formula in classical propositional logic
2. Euler Characteristic of a formula in Gödel logic
3. Euler Characteristic of a formula in Nilpotent Minimum logic
For an integer $n \geq 0$, let $\text{FORM}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\bot$ (falsum).
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- So we can consider valuations on \( \text{FORM}_n / \equiv \). In particular, let \( \chi \) be the Euler(-Klee-Rota) characteristic of \( \text{FORM}_n / \equiv \).
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- So we can consider valuations on $\text{FORM}_n/\equiv$. In particular, let $\chi$ be the Euler(-Klee-Rota) characteristic of $\text{FORM}_n/\equiv$.

- Then we say that the *Euler characteristic of* $\varphi$ *is* $\chi([\varphi]_{\equiv})$. 
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Euler characteristic of Boolean algebras

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- So if \( B \) is a finite Boolean algebra, and \( x \in B \) is the join of \( n \) atoms, we have \( \chi(x) = n \) by the valuation property. (The characteristic is additive over disjoint elements.)
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- In other words, if \( B \) is canonically represented as the Boolean algebra of all subsets of a set (=the set of atoms of \( B \)), then \( \chi(S) = \text{cardinality of } S \) for all \( S \in B \).
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- Since atoms of $\text{FORM}_n/\equiv$ are in natural bijections with assignments of truth values $\mu: \{X_1, \ldots, X_n\} \rightarrow \{0, 1\}$, we have:
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$\chi(\lbrack \varphi \rbrack_\equiv)$ is the number of assignments that satisfy $\varphi$. 
Gödel logic

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An assignment is a function $\mu: \text{FORM} \to [0, 1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha, \beta \in \text{FORM}$,

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\mu(\alpha \land \beta) = \min\{\mu(\alpha), \mu(\beta)\}
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and $\mu(\neg \alpha) = \mu(\alpha \rightarrow \bot), \mu(\bot) = 0, \mu(\top) = 1$. 

Gödel logic \( G_\infty \) can be semantically defined as a many-valued logic.

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An assignment is a function \( \mu : \text{FORM} \rightarrow [0, 1] \subseteq \mathbb{R} \) with values in the real unit interval such that, for any two \( \alpha, \beta \in \text{FORM} \),

\[
\begin{align*}
\mu(\alpha \land \beta) &= \min\{\mu(\alpha), \mu(\beta)\} \\
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A tautology is a formula \( \alpha \) such that \( \mu(\alpha) = 1 \) for every assignment \( \mu \).
Gödel algebras

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity axiom

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\[(x \to y) \lor (y \to x) = \top.\]

They provide the equivalent algebraic semantics of Gödel logic. For an integer \(n \geq 0\), let us write \(G_n\) for the Tarski-Lindenbaum algebra of Gödel logic over the variables \(X_1, \ldots, X_n\), that is, the algebra \(\text{FORM}_n/\equiv\), where \(\equiv\) is the logical equivalence between formulæ.
Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \text{FORM}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_{\equiv})$, where $\chi$ is the Euler characteristic of the finite distributive lattice $G_n$. 
The Euler characteristic of a formula \( \varphi \in \text{FORM}_n \), written \( \chi(\varphi) \), is the number \( \chi([\varphi]_{\equiv}) \), where \( \chi \) is the Euler characteristic of the finite distributive lattice \( G_n \).

Does this notion have any logical content?
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**Theorem**

Fix an integer $n \geq 1$. For any formula $\varphi \in \text{FORM}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu: \text{FORM}_n \rightarrow [0, 1]$ such that $\mu(\varphi) = 1$. 

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In the sense given by this result, the characteristic of a formula as defined above is a classical notion – it will not distinguish, for instance, classical from non-classical tautologies.
Gödel \((k + 1)\)-valued logic

We shall use Gödel \((k + 1)\)-valued logic, written \(\mathbb{G}_{k+1}\), for an integer \(k \geq 1\).
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\(\mathcal{G}_{k+1}\) is obtained from \(\mathcal{G}_\infty\), Gödel (infinite-valued) logic recalled above, by restricting assignments to those taking values in the set

\[ V_{k+1} = \{0 = \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \frac{k}{k} = 1\} \subseteq [0, 1] , \]

that is, to \((k + 1)\)-valued assignments.
Generalised Euler characteristic of a formula in Gödel logic

For a join-irreducible $g \in \mathcal{G}_n$, say $g$ has height $h(g)$ if the (unique) chain of join-irreducibles below $g$ in $\mathcal{G}_n$ has cardinality $h(g)$. 
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**Generalised Euler characteristic**

Fix integers $n, k \geq 1$. We write $\chi_k : G_n \rightarrow \mathbb{R}$ for the unique valuation on $G_n$ that satisfies

$$\chi_k(g) = \min \{ h(g), k \}$$

for each join-irreducible element $g \in G_n$. Further, if $\varphi \in \text{FORM}_n$, we define $\chi_k(\varphi) = \chi_k([\varphi]_\equiv)$.

It turns out that $\chi_k$ is a “$k$-valued characteristic”, as we proceed to show.
Our next aim is to relate $\chi_k$ with (not necessarily Boolean) $[0, 1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu: \{X_1\} \to [0, 1]$. However, in Gödel logic this fact is quite misleading, and there is the following important reduction to finiteness.
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$n$-equivalence

Fix integers $n, k \geq 1$. We say that two $(k+1)$-valued assignments $\mu$ and $\nu$ are equivalent over the first $n$ variables, or just $n$-equivalent, if and only if for all formulæ $\varphi(X_1, \ldots, X_n)$ of $\mathbb{G}_{k+1}$, $\mu(\varphi) = 1$ if and only if $\nu(\varphi) = 1$. The same definition can be given, mutatis mutandis, for $\mathbb{G}_\infty$. 
Reduction to finitely many possible worlds

In $G_\infty$, there are only finitely many equivalence classes of $[0, 1]$-valued assignments to $n$ variables.
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where

$$P(n, k) = \sum_{i=1}^{k} \sum_{j=0}^{n} \binom{n}{j} T(j, i),$$
and

$$T(n, k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > n + 1, \\ \sum_{i=1}^{n} \binom{n}{i} T(n - i, k - 1) & \text{otherwise}. \end{cases}$$
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The number \(P(n, k)\) of distinct equivalence classes of \((k + 1)\)-valued assignments over \(n\) variables.
Main result

**Theorem**

Fix integers \( n, k \geq 1 \), and a formula \( \varphi \in \text{FORM}_n \).

\[ \chi_k(\varphi) \text{ equals the number of } (k + 1)-\text{valued assignments } \mu: \text{FORM}_n \rightarrow [0, 1] \text{ such that } \mu(\varphi) = 1, \text{ up to } n\text{-equivalence.} \]
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1. $\chi_k(\varphi)$ equals the number of $(k + 1)$-valued assignments $\mu : \text{FORM}_n \to [0, 1]$ such that $\mu(\varphi) = 1$, up to $n$-equivalence.

2. $\varphi$ is a tautology in $\mathbb{G}_{k+1}$ if and only if $\chi_k(\varphi) = P(n, k)$. 
Main result

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2. $\varphi$ is a tautology in $G_{k+1}$ if and only if $\chi_k(\varphi) = P(n, k)$.

3. $\varphi$ is a tautology in $G_{\infty}$ if and only if it is a tautology in $G_{n+2}$ if and only if $\chi_{n+1}(\varphi) = P(n, n + 1)$. 
Proof of main result

**Lemma 1**

Fix integers \( n, k \geq 1 \), let \( x \in G_n \) and consider the valuation \( \chi_k : G_n \rightarrow \mathbb{R} \). Then, \( \chi_k(x) \) equals the number of join-irreducible elements \( g \in G_n \) such that \( g \leq x \) and \( h(g) \leq k \).
Proof of main result

**Lemma 1**

Fix integers $n, k \geq 1$, let $x \in \mathcal{G}_n$ and consider the valuation $\chi_k : \mathcal{G}_n \to \mathbb{R}$. Then, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq x$ and $h(g) \leq k$.

**Lemma 2**

Fix integers $n, k \geq 1$, and let $\varphi \in \text{FORM}_n$. Let $O(\varphi, n, k)$ be the set of equivalence classes $[\mu]_{\equiv^k_n}$ of $(k+1)$-valued assignments such that $\mu(\varphi) = 1$. Further, let $J(\varphi, n, k)$ be the set of join-irreducible elements $x \in \mathcal{G}_n$ such that $x \leq [\varphi]_{\equiv}$ and $h(x) \leq k$. Then there is a bijection between $O(\varphi, n, k)$ and $J(\varphi, n, k)$. 
Example

The Gödel algebra $G_1$, and the values of $\chi = \chi_1 : G_1 \rightarrow \mathbb{R}$ and $\chi_2 : G_1 \rightarrow \mathbb{R}$. 
## Nilpotent Minimum logic

**NM logic** $NM$ can be semantically defined as a many-valued logic.
Nilpotent Minimum logic

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Let \( \text{FORM} \) be the set of formulæ over propositional variables \( X_1, X_2, \ldots \) in the language \( \odot, \land, \lor, \to, \neg, \bot, \top \).
Nilpotent Minimum logic

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Let \(\text{FORM}\) be the set of formulæ over propositional variables \(X_1, X_2, \ldots\) in the language \(\odot, \land, \lor, \rightarrow, \neg, \bot, \top\).

An assignment is a function \(\mu: \text{FORM} \rightarrow [0, 1] \subseteq \mathbb{R}\) with values in the real unit interval such that, for any two \(\alpha, \beta \in \text{FORM}\),

\[
\mu(\alpha \odot \beta) = \begin{cases} 
\min\{\mu(\alpha), \mu(\beta)\} & \text{if } \mu(\alpha) + \mu(\beta) > 1 \\
0 & \text{otherwise}
\end{cases}
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\mu(\alpha \land \beta) = \min\{\mu(\alpha), \mu(\beta)\}
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and \(\mu(\neg \alpha) = 1 - \mu(\alpha), \mu(\bot) = 0, \mu(\top) = 1\).
**Nilpotent Minimum logic**

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Let $\text{FORM}$ be the set of formulæ over propositional variables $X_1, X_2, \ldots$ in the language $\odot, \land, \lor, \to, \neg, \bot, \top$.

An **assignment** is a function $\mu : \text{FORM} \to [0, 1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha, \beta \in \text{FORM}$,

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A **tautology** is a formula $\alpha$ such that $\mu(\alpha) = 1$ for every assignment $\mu$. 
NM algebras are Nelson algebras satisfying the prelinearity axiom

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They provide the equivalent algebraic semantics of NM logic. For an integer \(n \geq 0\), let us write \(\text{NM}_n\) for the Tarski-Lindenbaum algebra of NM logic over the variables \(X_1, \ldots, X_n\), that is, the algebra \(\text{FORM}_n/\equiv\), where \(\equiv\) is the logical equivalence between formulæ.
The Euler characteristic of a formula $\varphi \in \text{FORM}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_{\equiv})$, where $\chi$ is the Euler characteristic of the finite distributive lattice $\text{NM}_n$. We can now hope that the Euler characteristic of a formula $\varphi$ can encode logical information similar to that encoded by the characteristic in the case of Gödel logic. Unfortunately, this is not the case. Indeed, take, for instance, the formula $\alpha = (X \leftrightarrow X)^2 \land X$. It turns out that: for every assignments $\mu : \text{FORM}_n \to [0, 1]$, $\mu(\alpha) < 1$, but $\alpha \equiv$ is a join irreducible and thus $\chi(\alpha) = 1$. 
Euler characteristic of a formula NM logic

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It turns out that:

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Positive Euler characteristic of a formula in NM logic

For an element $x \in \mathcal{NM}_n$, say $x$ is idempotent if $x \circ x = x$. 
Positive Euler characteristic of a formula in NM logic

For an element $x \in \mathcal{NM}_n$, say $x$ is idempotent if $x \odot x = x$.

Positive Euler characteristic

We write $\chi^+: \mathcal{NM}_n \to \mathbb{R}$ for the unique valuation on $\mathcal{NM}_n$ that satisfies:

1. $\chi^+(x) = 1$ for each idempotent join irreducible element $x \in \mathcal{NM}_n$.
2. $\chi^+(x \odot x) = \chi^+(x)$ for each $x \in \mathcal{NM}_n$;

Further, if $\varphi \in \text{FORM}_n$, we define $\chi^+(\varphi) = \chi^+(\lbrack \varphi \rbrack_\equiv)$. 
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**Positive Euler characteristic**

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Further, if $\varphi \in \text{FORM}_n$, we define $\chi^+(\varphi) = \chi^+([\varphi]_\equiv)$.

Note that $\alpha \odot \alpha = \bot$, thus $\chi^+(\alpha) = 0$. 
Main result

**Theorem**

Fix $n \geq 1$, and a formula $\varphi \in \text{FORM}_n$.

$\chi^+(\varphi)$ equals the number of assignments $\mu: \text{FORM}_n \rightarrow \{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$. 
Main result

**Theorem**
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$\chi^+(\varphi)$ equals the number of assignments $\mu : \text{FORM}_n \to \{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$.

Remark. If $\varphi$ is a tautology in $\text{NM}$, then $\chi^+(\varphi) = 3^n$. 
Proof of main result

**Lemma 1**
Fix $n \geq 1$. Let $x \in \mathcal{NM}_n$ and consider the valuation $\chi^+: \mathcal{NM}_n \to \mathbb{R}$. Then, $\chi^+(x)$ equals the number of minimal idempotent join-irreducible elements $g \in \mathcal{NM}_n$ such that $g \leq x$. 
Proof of main result

**Lemma 1**

Fix $n \geq 1$. Let $x \in \mathcal{NM}_n$ and consider the valuation $\chi^+: \mathcal{NM}_n \rightarrow \mathbb{R}$. Then, $\chi^+(x)$ equals the number of minimal idempotent join-irreducible elements $g \in \mathcal{NM}_n$ such that $g \leq x$.

**Lemma 2**

Fix $n \geq 1$, and let $\varphi \in \text{FORM}_n$. There is a bijection between the set of equivalence classes $[\mu]_{\equiv_n}$ of assignments to $\{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$ and the set of idempotent join-irreducible elements $x \in \mathcal{NM}_n$ such that $x \leq [\varphi]_{\equiv_n}$. 
Further research

- Investigate a *generalised* positive Euler characteristic for NM logic, as done for Gödel logic.
- Investigate the logical content of the Euler characteristic in NM logic.
- Investigate the Euler characteristic in $\text{NM}^-$. 
References

Euler Characteristic

Gödel Logic

NM Logic
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Thank you for your attention.