Products in the category of forests and p-morphisms via Delannoy paths on Cartesian products

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(joint work with Ottavio M. D'Antona, and Vincenzo Marra)

TACL 2015, Ischia (NA) — June 25, 2015

Basic notions.

A category of forests

- A forest is a finite poset F such that for every $x \in F$, $\downarrow x$ is a chain. A tree is a forest with a bottom element.

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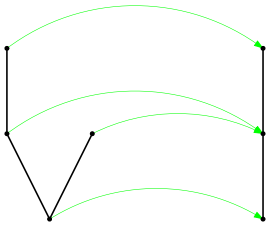

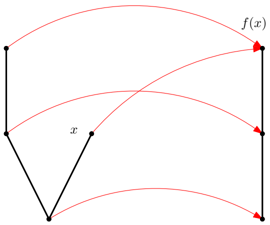

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Various techniques to perform this computation are known. Why should the one presented here be interesting?

- In the category F products are not Cartesian.
- Our construction is “as Cartesian as possible”.

Product of forests,
known combinatorial methods.

Ordered partitions, and merged shuffles

[D'Antona, O.M., and Marra, V., *Computing coproducts of finitely presented Gödel algebras*, Ann. Pure Appl. Logic 142 (2006), 202–211]

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- Let σ and τ be ordered partitions with disjoint supports. An ordered partition θ is a shuffle of σ and τ iff σ and τ are subsequences of θ, and $\mathbf{supp}\theta = \mathbf{supp}\sigma \cup \mathbf{supp}\tau$.

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Example. Let $\sigma = \{a|b\}$ and $\tau = \{x\}$. The merged shuffles of σ and τ are: $\{a|b|x\}$, $\{a|x|b\}$, $\{x|a|b\}$, $\{a|bx\}$, $\{ax|b\}$.

Trees of ordered partitions

Given ordered partitions $\sigma = \{A_1|\ldots|A_m\}$, and $\tau = \{B_1|\ldots|B_n\}$ with $m \leq n$ we write $\sigma \leq \tau$ iff $A_i = B_i$ for every $i \in \{1,\ldots,m\}$.

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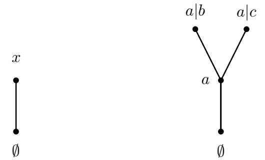

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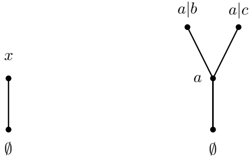

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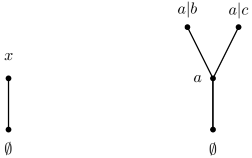

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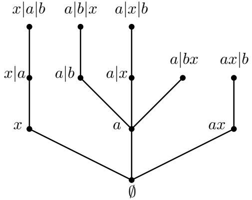

Product of forests

- Let $F = \{T_1, \dots, T_r\}$ and $G = \{U_1, \dots, U_s\}$ be forests.
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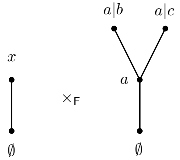
How to compute the product of trees?

Product of trees

Computing the product of trees (an example).



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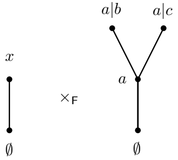

MERGED SHUFFLES

$ax\|b$	$a\|c\|x$
$ax\|c$	$a\|x\|b$
$a\|bx$	$a\|x\|c$
$a\|cx$	$x\|a\|b$
$a\|b\|x$	$x\|a\|c$

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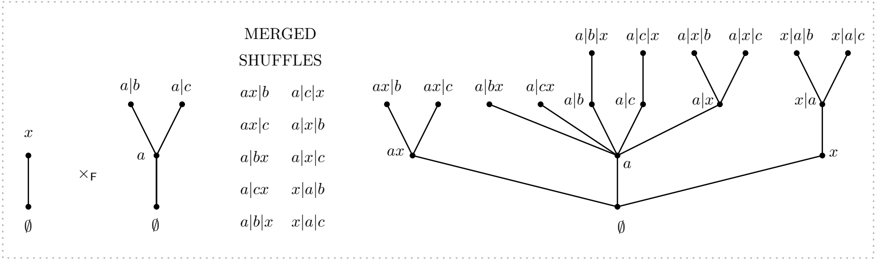

Product of forests, a recursive construction

[Aguzzoli, S., Bova, S., and Gerla, B., *Free Algebra and Functional Representation for Fuzzy Logics*, in Handbook of Mathematical Fuzzy Logic - Vol. 2, P. Cintula, P. Hájek, C. Noguera, eds., Studies in Logic, Vol. 38, College Pubblications, London (2011), 713–791]

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- If $|F| = 1$, then $F \times_{\mathsf{F}} G \cong G$.
- $(F + G) \times_{\mathsf{F}} H \cong (F \times_{\mathsf{F}} H) + (G \times_{\mathsf{F}} H)$.

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- $F_{\perp} \times_{\mathsf{F}} G_{\perp} \cong ((F \times_{\mathsf{F}} G_{\perp}) + (F \times_{\mathsf{F}} G) + (F_{\perp} \times_{\mathsf{F}} G))_{\perp}$.

Product of trees, a recursive formula

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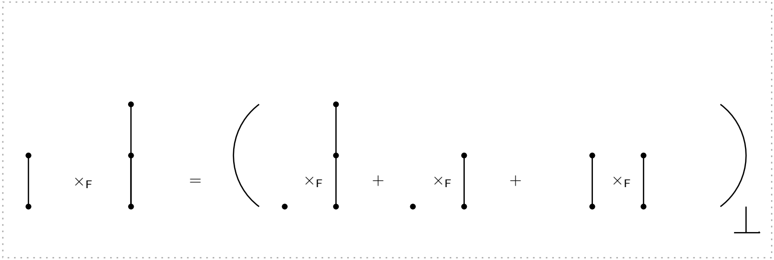

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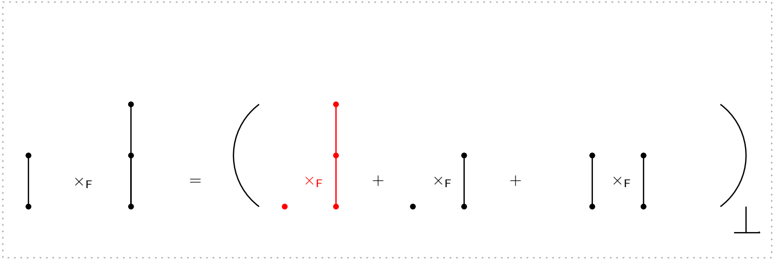

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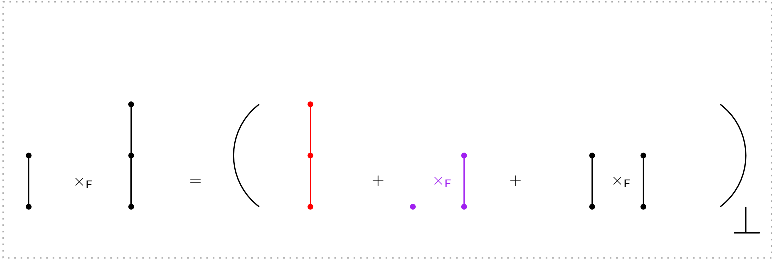

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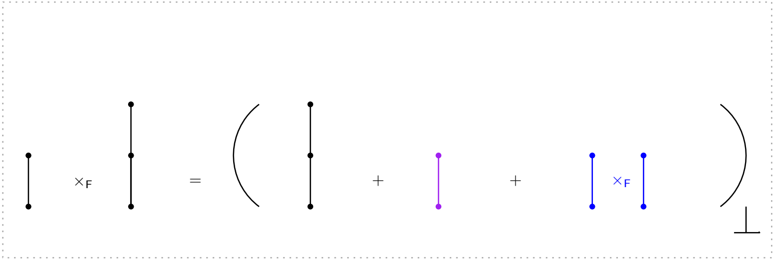

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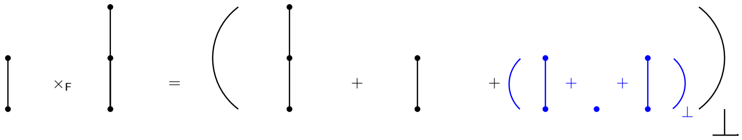

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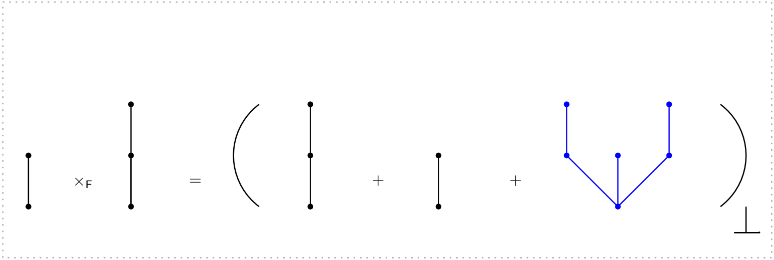

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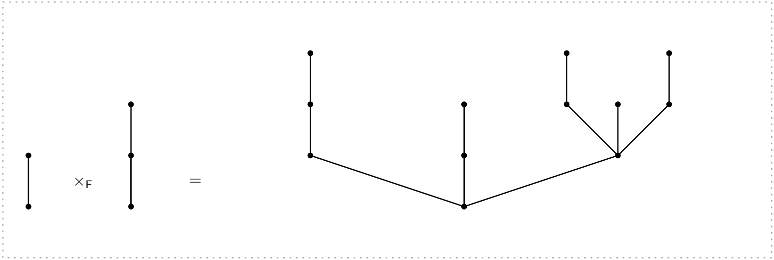

Products of forests via Delannoy paths on Cartesian products.

Classical Delannoy paths, and paths on posets

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- A (finite) path on a poset P is a non-empty sequence $\langle p_1, p_2, \ldots, p_h \rangle$ of elements of P such that $p_i < p_j$ whenever $i < j$. (A path on P is therefore the same thing as a chain of P.)

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- For each $i \in \{1, \ldots, n-1\}$, the pair p_i, p_{i+1} is called a step of the path.
- Given a poset P, and two elements $p, q \in P$, we write $p \lessdot q$ to indicate that q covers p in P, that is, $p < q$ and for every $s \in P$, if $p \leq s \leq q$, then either $s = p$ or $s = q$.

Delannoy paths on Cartesian products of posets

Definition

Let $P_1, \ldots, P_n$ be posets, and let $P = P_1 \times \cdots \times P_n$ be their (Cartesian) product. Let $\langle p_1, \ldots, p_h \rangle$ be a path on P.

The step from $p_i = (p_{i,1}, \ldots, p_{i,n})$ to $p_{i+1} = (p_{i+1,1}, \ldots, p_{i+1,n})$ is a Delannoy step, written $p_i \prec p_{i+1}$, if and only if there exists $k \in \{1, \ldots, n\}$ such that $p_{i,k} \neq p_{i+1,k}$, and for each $j \in \{1, \ldots, n\}$, $p_{i,j} \trianglelefteq p_{i+1,j}$.

The path $\langle p_1, \ldots, p_h \rangle$ on P is a Delannoy path if and only if p_1 is a minimal element of P, and for each $i \in \{1, \ldots, n-1\}$, $p_i \prec p_{i+1}$.

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- Delannoy paths on a poset $P = P_1 \times \cdots \times P_n$ can be partially ordered by $\langle q_1, \ldots, q_m \rangle \leq \langle p_1, \ldots, p_h \rangle$ if and only if $m \leq h$ and $q_i = p_i$ for each $i \in \{1, \ldots, m\}$.

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- We denote by $\mathcal{D}(P_1, \ldots, P_n)$ the poset of all Delannoy paths on P.
- Clearly, $\mathcal{D}(P_1, \ldots, P_n)$ is a forest.

Product in F via Delannoy paths

Theorem

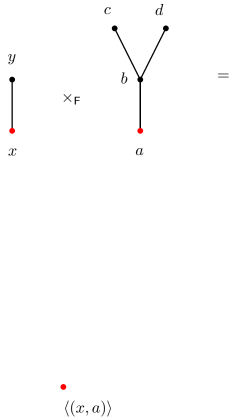
Let F and G be forests. Then $\mathcal{D}(F, G)$ is the product $F \times_{\mathsf{F}} G$ in the category F:

$$F \xleftarrow{\pi_F} F \times_{\mathsf{F}} G \xrightarrow{\pi_G} G\,.$$

Let $d \in F \times_{\mathsf{F}} G$, with $d = \langle (f_1, g_1), \ldots, (f_n, g_n) \rangle$. The projection functions $\pi_F : F \times_{\mathsf{F}} G \to F$ and $\pi_G : F \times_{\mathsf{F}} G \to G$ are defined by

$$\pi_F(d) = f_n, \text{ and } \ \pi_G(d) = g_n\,.$$

Computing the product in F, an example



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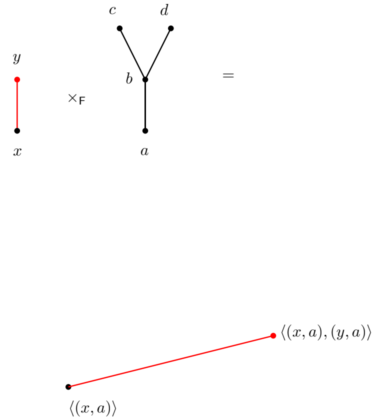

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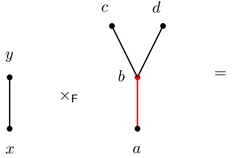

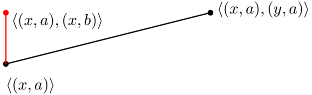

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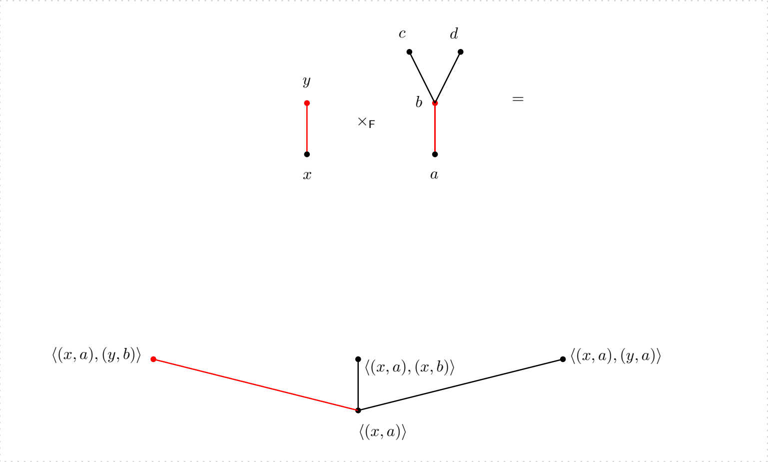

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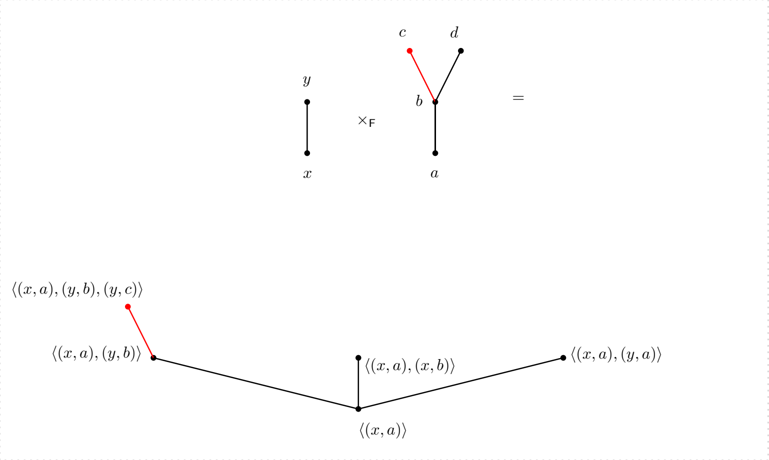

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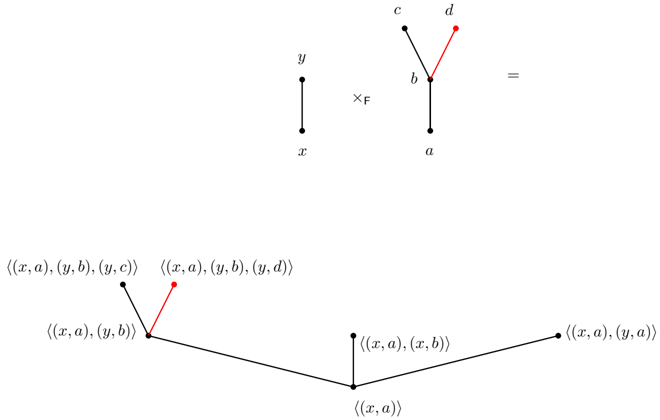

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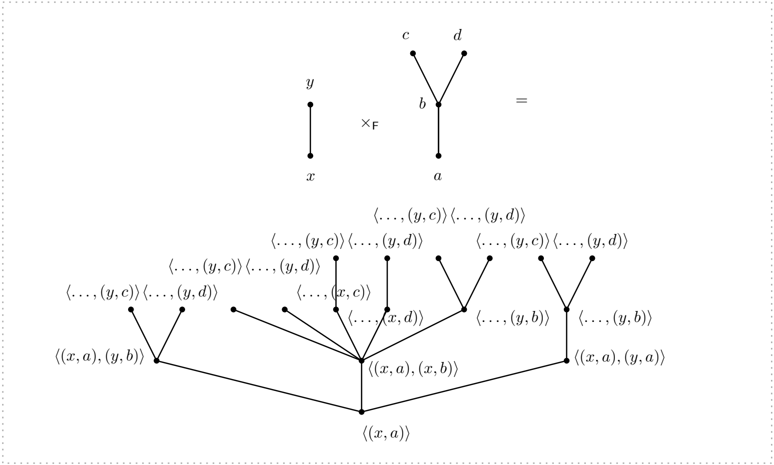

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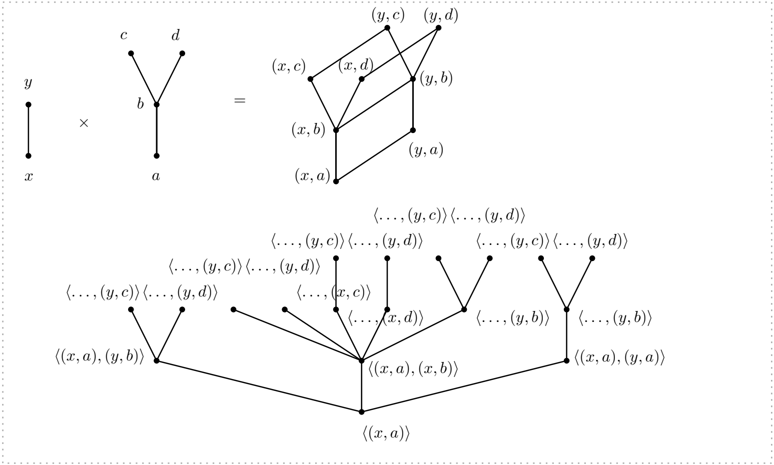

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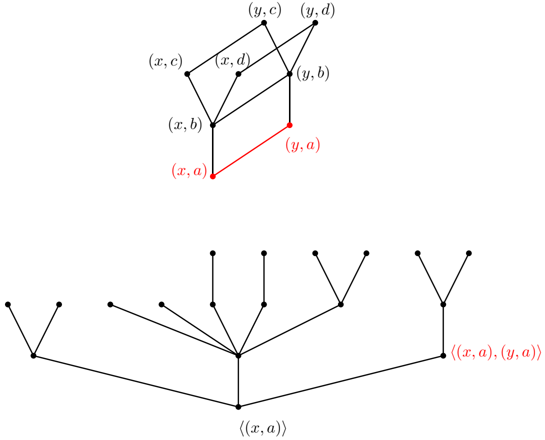

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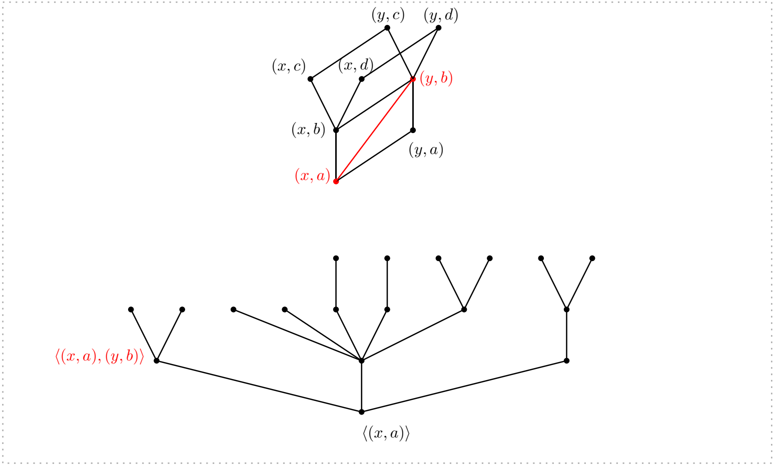

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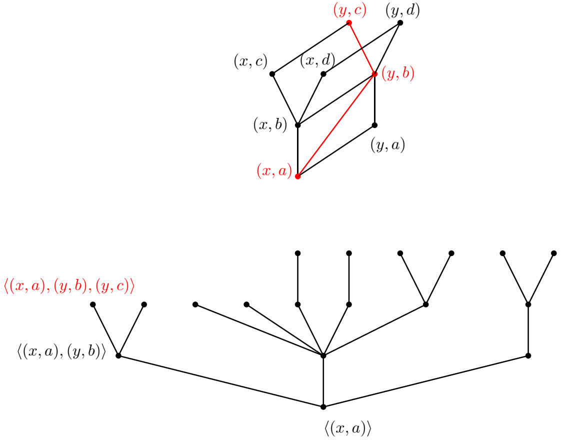

Enumeration.

Delannoy numbers

The Delannoy number $D_{n,m}$ counts the number of Delannoy paths from $(0,0)$ to (n,m). Delannoy numbers satisfy the following recurrence relation.

$$D_{n,m} = D_{n-1,m} + D_{n,m-1} + D_{n-1,m-1}$$

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The following table shows some values of Delannoy numbers.

1	1	1	1	1	1	1	1
1	3	5	7	9	11	13	15
1	5	13	25	41	61	85	113
1	7	25	63	129	231	377	575
1	9	41	129	321	681	1289	2241
1	11	61	231	681	1683	3653	7183

A Formula for the number of elements of the products

Let T, U be trees.

$$|T \times_{\mathsf{F}} U| = \sum_{i \geq 0} \sum_{j \geq 0} t_i u_j D_{i,j} \ ,$$

where t_i is the number of elements at level i of T, and u_j is the number of elements at level j of U.

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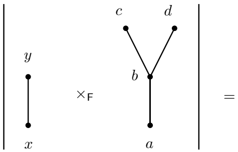
Example.



$= 1 \cdot 1 \cdot D_{0,0} + 1 \cdot 1 \cdot D_{0,1} + 1 \cdot 2 \cdot D_{0,2} + 1 \cdot 1 \cdot D_{1,0} + 1 \cdot 1 \cdot D_{1,1} + 1 \cdot 2 \cdot D_{1,2} =$
$= 1 + 1 + 2 + 1 + 3 + 10 = 18.$

A Formula for the number of elements of the products

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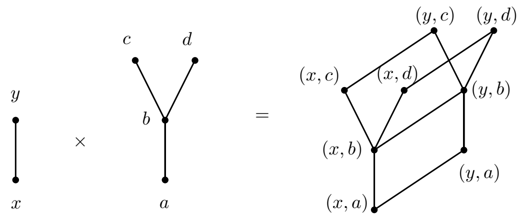

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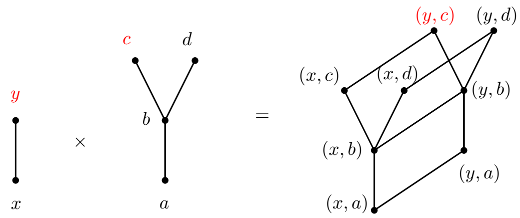

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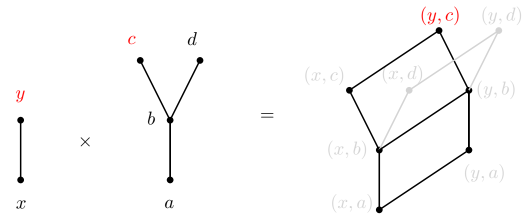

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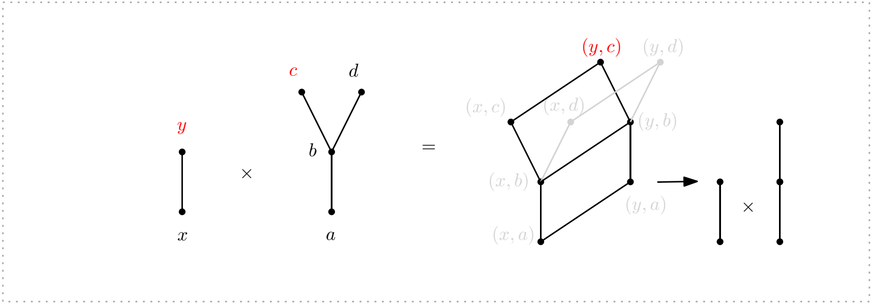
$$|T \times_{\mathsf{F}} U| = \sum_{i \geq 0} \sum_{j \geq 0} t_i u_j D_{i,j}$$



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Thank you for your attention.