Profinite Heyting Algebras, and Partitions of Image-Finite Posets under Open Maps

Pietro Codara, Vincenzo Marra
Dipartimento di Informatica e Comunicazione
Università degli Studi di Milano

Presenting author: Pietro Codara

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Aim

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In this talk we present a first case study. We obtain the appropriate notion of partition for the objects of the category that is dual to profinite Heyting algebras (i.e., Heyting algebras that can be represented as an inverse limit of an inverse family of finite Heyting algebras).

Outline

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- Related works and conclusion

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By the notion of epimorphism, we can define a partition as follows.

- A partition of the set A is the set $\{f^{-1}(b) \mid b \in B\}$ of *fibres* of an epimorphism (i.e., a surjection) $f : A \to B$.

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Traditionally, one does away with the arrow f, and directly defines a partition of a set A.

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2. **In terms of equivalent relations**: a partition of a set A is the set of equivalence classes of an equivalence relation on A.

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In this work we focus on the first characterisation. We will thus present a charactesation of a partition in ImOPos analogous to the first one.

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- P is called *image-finite* if $\downarrow S$ is finite whenever S is finite.

Open maps

- An order preserving function $f : P \to Q$ between posets is called *open* if whenever $f(u) \geqslant v'$ for some $u \in P$ and $v' \in Q$, there is $v \in P$ such that $u \geqslant v$ and $f(v) = v'$.

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Open maps are also known as *bounded morphisms*, or *p-morphisms*.

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In *G. Bezhanishvili, and N. Bezhanishvili, Profinite Heyting algebras, Order, 25, 2008, 3, 211–227,*

Guram and Nick Bezhanishvili prove that

- ImOPos is dually equivalent to the category ProHA of Profinite Heyting Algebras and their complete homomorphisms.

Open partitions as fibres

Definition

An *open partition* of an image-finite poset P is a set-theoretic partition $\pi = \{B_q \mid q \in Q\}$ of P that is induced by some surjective open map $f\colon P \to Q$ onto an image-finite poset Q. That is, for each $q \in Q$,

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Note that π, regarded as a poset under $\preceq$, is an isomorphic copy of Q.

Characterisation of open partitions

Theorem (Characterisation of open partitions)

Let P be an image-finite poset, and let $\pi = \{B_i \mid i \in I\}$ be a set-theoretic partition of P, where I is some index set. Then π is an open partition of P if and only if for each $B_i \in \pi$ there exist a subset $J \subseteq I$ such that

$$\uparrow B_i = \bigcup_{j \in J} B_j \,. \tag{1}$$

In this case, the underlying order $\preceq$ of π is uniquely determined by

$$B_i \preceq B_j \text{ iff } B_j \subseteq \uparrow B_i \text{ iff there are } x \in B_i, y \in B_j \text{ with } x \leqslant y\,,$$

for each $B_i, B_j \in \pi$.

Example

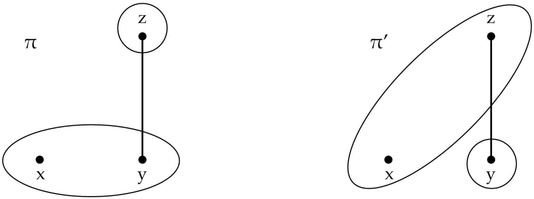

Figure: Two set-theoretic partitions of a poset. While π is open, π' is not.

Proof

One interesting point of the proof is to show that $\preceq$ is a partial order on π. Recall that

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Let $B_i, B_j \in \pi$ be such that $B_i \preceq B_j$ and $B_j \preceq B_i$. Let $x \in B_i$. Since $B_j \preceq B_i$ there exists $y \in B_j$ such that $y \leqslant x$. Since $B_i \preceq B_j$ there exists $z \in B_i$ such that $z \leqslant y \leqslant x$.

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Remarks on the infinite case

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Thus, the set-theoretic partition π satisfies Condition (1) in the Theorem.

But, clearly, we cannot construct any order preserving surjection having π as set of fibres.

Previous work

Clearly, the characterisation of partitions presented here also works in the finite case, that is, in the category $\mathtt{OPos}_{\mathtt{f}}$ of finite posets and open maps.

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Such a characterisation has been proved in

P. Codara, O. D'Antona, V. Marra: Open Partitions and Probability Assignments in Gödel Logic. In: ECSQARU 2009, LNCS (LNAI), vol. 5590, pp. 911–922. Springer, Heidelberg (2009),

where our result on partitions has been applied to the problem of developing an analogue of probability theory in Gödel logic.

Future work

- The case study presented in this talk is just a first step in the direction of extending such a ‘nice’ characterisation of partitions to categories having the same ‘good properties’ as the category $\mathtt{OPos}_{\mathtt{f}}$.

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- Future work: using the same approach, investigating the notion of partition in other *topoi*.

Related works

- As mentioned before, in ImOPos arrows factorize in an essentially unique way as an epimorphism followed by a monomorphism. This fails in the category $\mathtt{Pos_f}$ of finite poset and order preserving maps.

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- In $\mathtt{Pos_f}$ both (epi, regular mono) and (regular epi,mono) are factorisation systems, and the classes of epis and regular epis do not coincide.
- In this case, two different notions of partition will arise by taking fibres of epis and regular epis, respectively.

Related works

A combinatorial analysis of the two different notions of partitions in the category $\mathtt{Pos_f}$ can be found in

P. Codara, A theory of partitions of partially ordered sets, Ph.D. thesis, Università degli Studi di Milano, Italy, 2008,

and in

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Here, the notions of partitions are given both in terms of blocks and in terms of reflexive and transitive relations (quasiorders).

Thank you

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