Valuations in Nilpotent Minimum Logic

Pietro Codara

Dipartimento di Informatica, Università degli Studi di Milano

(Joint work with Diego Valota)

ISMVL 2015, Waterloo — May 18$^{\text{th}}$, 2015

The Euler-Klee-Rota lattice-theoretic characteristic

Valuation

Let L be a (bounded) distributive lattice whose bottom element is denoted $\perp$. A function $\nu\colon L \to \mathbb{R}$ is a valuation if it satisfies $\nu(\perp) = 0$, and

$$\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y)$$

for all $x, y \in L$.

The Euler-Klee-Rota lattice-theoretic characteristic

Valuation

Let L be a (bounded) distributive lattice whose bottom element is denoted $\bot$. A function $\nu\colon L \to \mathbb{R}$ is a valuation if it satisfies $\nu(\bot) = 0$, and

$$\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y)$$

for all $x, y \in L$.

Lemma

Every valuation on a finite distributive lattice L is uniquely determined by its values at the join-irreducibles of L.

Recall that $x \in L$ is *join-irreducible* if it is not the bottom of L, and $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$.

The Euler-Klee-Rota lattice-theoretic characteristic, definition

(V. Klee 1963; G.-C. Rota 1974)

Euler characteristic

The Euler characteristic of a finite distributive lattice L is the unique valuation $\chi\colon L \to \mathbb{R}$ such that $\chi(x) = 1$ for any join-irreducible element $x \in L$.

The Euler-Klee-Rota lattice-theoretic characteristic

- Let V be a set of *vertices*, and let P be the poset of subsets of V ordered by inclusion. The collection $\mathscr{L}$ of lower sets of P is a (bounded) distributive lattice under $\cap$, $\cup$.

The Euler-Klee-Rota lattice-theoretic characteristic

- Let V be a set of *vertices*, and let P be the poset of subsets of V ordered by inclusion. The collection $\mathscr{L}$ of lower sets of P is a (bounded) distributive lattice under $\cap$, $\cup$.
- An element $\Sigma \in \mathscr{L}$ is the same thing as a (combinatorial) simplicial complex: a collection of subsets of V such that $A \subseteq B \in \Sigma \Rightarrow A \in \Sigma$.

The Euler-Klee-Rota lattice-theoretic characteristic

- Let V be a set of *vertices*, and let P be the poset of subsets of V ordered by inclusion. The collection $\mathscr{L}$ of lower sets of P is a (bounded) distributive lattice under $\cap$, $\cup$.
- An element $\Sigma \in \mathscr{L}$ is the same thing as a (combinatorial) simplicial complex: a collection of subsets of V such that $A \subseteq B \in \Sigma \Rightarrow A \in \Sigma$.
- The Euler Characteristic on $\mathscr{L}$ is the unique valuation $\chi\colon \mathscr{L} \to \mathbb{R}$ such that $\chi(\emptyset) = 0$, and $\chi(\Delta) = 1$ whenever Δ is a simplex.

The Euler-Klee-Rota lattice-theoretic characteristic

- Let V be a set of *vertices*, and let P be the poset of subsets of V ordered by inclusion. The collection $\mathscr{L}$ of lower sets of P is a (bounded) distributive lattice under $\cap$, $\cup$.
- An element $\Sigma \in \mathscr{L}$ is the same thing as a (combinatorial) simplicial complex: a collection of subsets of V such that $A \subseteq B \in \Sigma \Rightarrow A \in \Sigma$.
- The Euler Characteristic on $\mathscr{L}$ is the unique valuation $\chi\colon \mathscr{L} \to \mathbb{R}$ such that $\chi(\emptyset) = 0$, and $\chi(\Delta) = 1$ whenever Δ is a simplex.
- It turns out that χ agrees with the classical Euler characteristic on each simplicial complex $\Sigma \in \mathscr{L}$.

Outline

1. Euler Characteristic of a formula in classical propositional logic
2. Euler Characteristic of a formula in Gödel logic
3. Euler Characteristic of a formula in Nilpotent Minimum logic

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\bot$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \textsc{Form}_n$?

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \textsc{Form}_n$?
- Writing $\equiv$ for the relation of logical equivalence, the quotient set $\textsc{Form}_n/\equiv$ is naturally a Boolean algebra.

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \textsc{Form}_n$?
- Writing $\equiv$ for the relation of logical equivalence, the quotient set $\textsc{Form}_n / \equiv$ is naturally a Boolean algebra.
- So we can consider valuations on $\textsc{Form}_n / \equiv$. In particular, let χ be the Euler(-Klee-Rota) characteristic of $\textsc{Form}_n / \equiv$.

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\bot$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \textsc{Form}_n$?
- Writing $\equiv$ for the relation of logical equivalence, the quotient set $\textsc{Form}_n/\equiv$ is naturally a Boolean algebra.
- So we can consider valuations on $\textsc{Form}_n/\equiv$. In particular, let χ be the Euler(-Klee-Rota) characteristic of $\textsc{Form}_n/\equiv$.
- Then we say that *the Euler characteristic of* φ *is* $\chi([\varphi]_\equiv)$.

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)
- In other words, if B is canonically represented as the Boolean algebra of all subsets of a set (=the set of atoms of B), then $\chi(S)$=cardinality of S for all $S \in B$.

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)
- In other words, if B is canonically represented as the Boolean algebra of all subsets of a set (=the set of atoms of B), then $\chi(S)$=cardinality of S for all $S \in B$.
- Since atoms of $\textsc{Form}_n/\equiv$ are in natural bijections with assignments of truth values $\mu\colon \{X_1, \ldots, X_n\} \to \{0, 1\}$, we have:

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)
- In other words, if B is canonically represented as the Boolean algebra of all subsets of a set (=the set of atoms of B), then $\chi(S)$=cardinality of S for all $S \in B$.
- Since atoms of $\textsc{Form}_n/\equiv$ are in natural bijections with assignments of truth values $\mu\colon \{X_1, \ldots, X_n\} \to \{0, 1\}$, we have:

$\chi([\varphi]_{\equiv})$ *is the number of assignments that satisfy* φ.

Gödel logic

Gödel logic $\mathbb{G}_\infty$ can be semantically defined as a many-valued logic.

Let $\textsc{Form}$ be the set of formulæ over propositional variables $X_1, X_2, \ldots$ in the language $\wedge, \vee, \rightarrow, \neg, \bot, \top$.

An assignment is a function $\mu\colon \textsc{Form} \rightarrow [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha,\ \beta \in \textsc{Form}$,

$$\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \vee \beta) = \max\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \rightarrow \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases}$$

and $\mu(\neg\alpha) = \mu(\alpha \rightarrow \bot)$, $\mu(\bot) = 0$, $\mu(\top) = 1$.

A tautology is a formula α such that $\mu(\alpha) = 1$ for every assignment μ.

Gödel algebras

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity axiom

$$(x \to y) \vee (y \to x) = \top \; .$$

They provide the equivalent algebraic semantics of Gödel logic. For an integer $n \geq 0$, let us write $\mathscr{G}_n$ for the Tarski-Lindenbaum algebra of Gödel logic over the variables $X_1, \ldots, X_n$, that is, the algebra $\textsc{Form}_n / \equiv$, where $\equiv$ is the logical equivalence between formulæ.

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Does this notion have any logical content?

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Does this notion have any logical content?

Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \textsc{Form}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$.

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Does this notion have any logical content?

Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \textsc{Form}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu \colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$.

In the sense given by this result, the characteristic of a formula as defined above is a classical notion – it will not distinguish, for instance, classical from non-classical tautologies.

Gödel $(k+1)$-valued logic

We shall use Gödel $(k+1)$-valued logic, written $\mathbb{G}_{k+1}$, for an integer $k \geq 1$.

Gödel $(k+1)$-valued logic

We shall use Gödel $(k+1)$-valued logic, written $\mathbb{G}_{k+1}$, for an integer $k \geq 1$.
$\mathbb{G}_{k+1}$ is obtained from $\mathbb{G}_{\infty}$, Gödel (infinite-valued) logic recalled above, by restricting assignments to those taking values in the set

$$V_{k+1} = \{0 = \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \frac{k}{k} = 1\} \subseteq [0,1] ,$$

that is, to $(k+1)$-*valued assignments.*

Generalised Euler characteristic of a formula in Gödel logic

For a join-irreducible $g \in \mathscr{G}_n$, say g has height $h(g)$ if the (unique) chain of join-irreducibles below g in $\mathscr{G}_n$ has cardinality $h(g)$.

Generalised Euler characteristic of a formula in Gödel logic

For a join-irreducible $g \in \mathscr{G}_n$, say g has height $h(g)$ if the (unique) chain of join-irreducibles below g in $\mathscr{G}_n$ has cardinality $h(g)$.

Generalised Euler characteristic

Fix integers $n, k \geq 1$. We write $\chi_k \colon \mathscr{G}_n \to \mathbb{R}$ for the unique valuation on $\mathscr{G}_n$ that satisfies

$$\chi_k(g) = \min\{h(g), k\}$$

for each join-irreducible element $g \in \mathscr{G}_n$. Further, if $\varphi \in \textsc{Form}_n$, we define $\chi_k(\varphi) = \chi_k([\varphi]_\equiv)$.

It turns out that χ_k is a "k-valued characteristic", as we proceed to show.

n-equivalence

Our next aim is to relate χ_k with (not necessarily Boolean) $[0,1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu\colon \{X_1\} \to [0,1]$. However, in Gödel logic there is the following important reduction to finiteness.

n-equivalence

Our next aim is to relate χ_k with (not necessarily Boolean) $[0,1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu\colon \{X_1\} \to [0,1]$. However, in Gödel logic there is the following important reduction to finiteness.

n-equivalence

Fix integers $n, k \geq 1$. We say that two $(k+1)$-valued assignments μ and ν are *equivalent over the first n variables*, or just *n-equivalent*, if and only if for all formulæ $\varphi(X_1, \ldots, X_n)$ of $\mathbb{G}_{k+1}$, $\mu(\varphi) = 1$ if and only if $\nu(\varphi) = 1$.
The same definition can be given, *mutatis mutandis*, for $\mathbb{G}_\infty$.

n-equivalence

Our next aim is to relate χ_k with (not necessarily Boolean) $[0,1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu\colon \{X_1\} \to [0,1]$. However, in Gödel logic there is the following important reduction to finiteness.

n-equivalence

Fix integers $n, k \geq 1$. We say that two $(k+1)$-valued assignments μ and ν are *equivalent over the first n variables*, or just *n-equivalent*, if and only if for all formulæ $\varphi(X_1, \ldots, X_n)$ of $\mathbb{G}_{k+1}$, $\mu(\varphi) = 1$ if and only if $\nu(\varphi) = 1$. The same definition can be given, *mutatis mutandis*, for $\mathbb{G}_\infty$.

In $\mathscr{G}_\infty$, there are only finitely many equivalence classes of $[0,1]$-valued assignments to n variables.

Main result

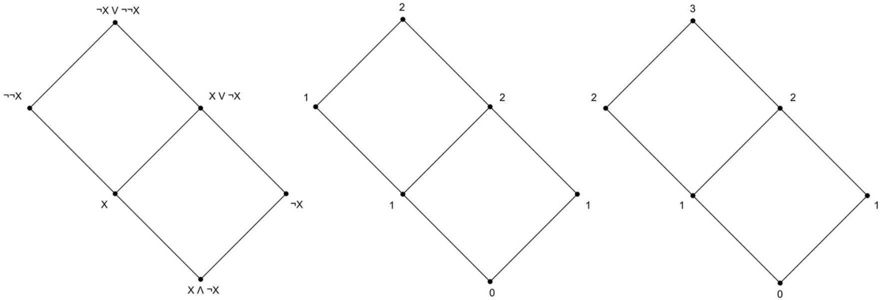
Theorem

Fix integers $n, k \geq 1$, and a formula $\varphi \in \textsc{Form}_n$.

> $\chi_k(\varphi)$ equals the number of $(k+1)$-valued assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$, up to n-equivalence.

Moreover, φ is a tautology in $\mathbb{G}_\infty$ if and only if it is a tautology in $\mathbb{G}_{n+2}$.

Example



The Gödel algebra $\mathscr{G}_1$, and the values of $\chi = \chi_1 : \mathscr{G}_1 \to \mathbb{R}$ and $\chi_2 : \mathscr{G}_1 \to \mathbb{R}$.

Nilpotent Minimum logic

NM logic $\mathbb{NM}$ can be semantically defined as a many-valued logic.

Let FORM be the set of formulæ over propositional variables $X_1, X_2, \ldots$ in the language $\odot, \wedge, \vee, \rightarrow, \neg, \bot, \top$.

An assignment is a function $\mu\colon \text{FORM} \rightarrow [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha,\ \beta \in \text{FORM}$,

$$\mu(\alpha \odot \beta) = \begin{cases} \min\{\mu(\alpha), \mu(\beta)\} & \text{if } \mu(\alpha) + \mu(\beta) > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \vee \beta) = \max\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \rightarrow \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \max\{1 - \mu(\alpha), \mu(\beta)\} & \text{otherwise} \end{cases}$$

and $\mu(\neg\alpha) = 1 - \mu(\alpha)$, $\mu(\bot) = 0$, $\mu(\top) = 1$.

A tautology is a formula α such that $\mu(\alpha) = 1$ for every assignment μ.

NM algebras

NM algebras are Nelson algebras satisfying the prelinearity axiom

$$(x \to y) \vee (y \to x) = \top \ .$$

They provide the equivalent algebraic semantics of NM logic. For an integer $n \geq 0$, let us write $\mathscr{NM}_n$ for the Tarski-Lindenbaum algebra of NM logic over the variables $X_1, \ldots, X_n$, that is, the algebra $\textsc{Form}_n / \equiv$, where $\equiv$ is the logical equivalence between formulæ.

Euler characteristic of a formula NM logic

Euler characteristic of a formula in NM logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_{\equiv})$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{NM}_n$.

Euler characteristic of a formula NM logic

Euler characteristic of a formula in NM logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{NM}_n$.

We can now hope that the Euler characteristic of a formula φ can encode logical information similar to that encoded by the characteristic in the case of Gödel logic.

Euler characteristic of a formula NM logic

Euler characteristic of a formula in NM logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{NM}_n$.

We can now hope that the Euler characteristic of a formula φ can encode logical information similar to that encoded by the characteristic in the case of Gödel logic.

Unfortunately, this is not the case. Indeed, take, for instance, the formula

$$\alpha = (X \leftrightarrow \neg X)^2 \wedge X$$

Euler characteristic of a formula NM logic

Euler characteristic of a formula in NM logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{NM}_n$.

We can now hope that the Euler characteristic of a formula φ can encode logical information similar to that encoded by the characteristic in the case of Gödel logic.
Unfortunately, this is not the case. Indeed, take, for instance, the formula

$$\alpha = (X \leftrightarrow \neg X)^2 \wedge X$$

It turns out that:

Euler characteristic of a formula NM logic

Euler characteristic of a formula in NM logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_{\equiv})$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{NM}_n$.

We can now hope that the Euler characteristic of a formula φ can encode logical information similar to that encoded by the characteristic in the case of Gödel logic.
Unfortunately, this is not the case. Indeed, take, for instance, the formula

$$\alpha = (X \leftrightarrow \neg X)^2 \wedge X$$

It turns out that:

- For every assignments $\mu\colon \textsc{Form}_n \to [0,1]$, $\mu(\alpha) < 1$, but

Euler characteristic of a formula NM logic

Euler characteristic of a formula in NM logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{NM}_n$.

We can now hope that the Euler characteristic of a formula φ can encode logical information similar to that encoded by the characteristic in the case of Gödel logic.
Unfortunately, this is not the case. Indeed, take, for instance, the formula

$$\alpha = (X \leftrightarrow \neg X)^2 \wedge X$$

It turns out that:

- For every assignments $\mu \colon \textsc{Form}_n \to [0,1]$, $\mu(\alpha) < 1$, but
- $[\alpha]_\equiv$ is a join irreducible and thus $\chi(\alpha) = 1$.

Idempotent Euler characteristic of a formula in NM logic

For an element $x \in \mathscr{NM}_n$, say x is idempotent if $x \odot x = x$.

Idempotent Euler characteristic of a formula in NM logic

For an element $x \in \mathscr{NM}_n$, say x is idempotent if $x \odot x = x$.

Idempotent Euler characteristic

We write $\chi^+ \colon \mathscr{NM}_n \to \mathbb{R}$ for the unique valuation on $\mathscr{NM}_n$ that satisfies:

1. $\chi^+(\bot) = 0$;
2. for each join irreducible element $x \in \mathscr{NM}_n$,
$$\chi^+(g) = \begin{cases} 1 & \text{if } g \odot g = g\,, \\ 0 & \text{otherwise.} \end{cases}$$

Further, if $\varphi \in \textsc{Form}_n$, we define $\chi^+(\varphi) = \chi^+([\varphi]_\equiv)$.

Idempotent Euler characteristic of a formula in NM logic

For an element $x \in \mathcal{NM}_n$, say x is idempotent if $x \odot x = x$.

Idempotent Euler characteristic

We write $\chi^+ \colon \mathcal{NM}_n \to \mathbb{R}$ for the unique valuation on $\mathcal{NM}_n$ that satisfies:

1. $\chi^+(\bot) = 0$;
2. for each join irreducible element $x \in \mathcal{NM}_n$,
$$\chi^+(g) = \begin{cases} 1 & \text{if } g \odot g = g\,, \\ 0 & \text{otherwise.} \end{cases}$$

Further, if $\varphi \in \textsc{Form}_n$, we define $\chi^+(\varphi) = \chi^+([\varphi]_\equiv)$.

Observe that, if g is a non-idempotent join irreducible element, then $g \odot g = \bot$.

Main result

Proposition

Fix $n \geq 1$. The idempotent Euler characteristic satisfies, for every $x \in \mathscr{NM}_n$,

$$\chi^+(x \odot x) = \chi^+(x)$$

Main result

Proposition

Fix $n \geq 1$. The idempotent Euler characteristic satisfies, for every $x \in \mathscr{NM}_n$,

$$\chi^+(x \odot x) = \chi^+(x)$$

Theorem

For any formula $\varphi \in \textsc{Form}_n$, the valuation $\chi^+(\varphi)$ equals the number of assignments $\mu\colon \textsc{Form}_n \to \{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$.

Main result

Proposition

Fix $n \geq 1$. The idempotent Euler characteristic satisfies, for every $x \in \mathscr{NM}_n$,

$$\chi^{+}(x \odot x) = \chi^{+}(x)$$

Theorem

For any formula $\varphi \in \textsc{Form}_n$, the valuation $\chi^{+}(\varphi)$ equals the number of assignments $\mu\colon \textsc{Form}_n \to \{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$.

Remark. If φ is a tautology in $\mathbb{NM}$, then $\chi^{+}(\varphi) = 3^n$.

Proof of main result

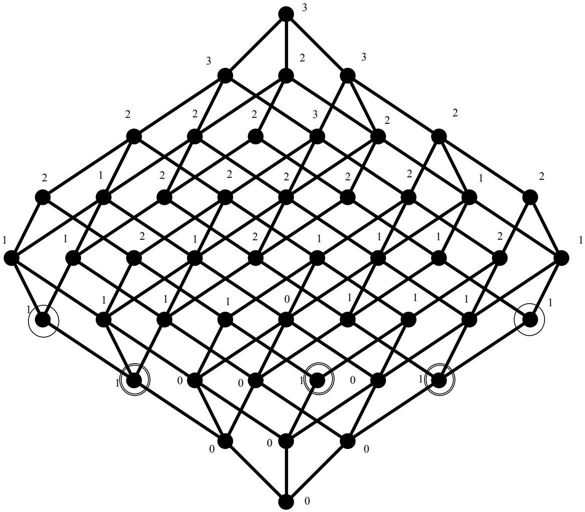
Lemma 1

Fix integer $n \geq 1$, and let $x \in \mathscr{NM}_n$. Then, $\chi^+(x)$ equals the number of minimal idempotent join-irreducible elements $g \in \mathscr{NM}_n$ such that $g \leq x$.

Lemma 2

Fix $n \geq 1$, and let $\varphi \in \textsc{Form}_n$. Let $O(\varphi, n)$ be the set of assignments $\mu : \textsc{Form}_n \to \{0, \frac{1}{2}, 1\}$ such that $\mu(\varphi) = 1$. Then, there is a bijection between $O(\varphi, n)$ and the set of minimal idempotent join irreducible elements $g \in \mathscr{NM}_n$ such that $g \leq [\varphi]_\equiv$.

Example



The Nilpotent Minimum algebra $\mathscr{NM}_1$, with the values of $\chi^+ \colon \mathscr{NM}_1 \to \mathbb{R}$.

NM^- logic

NM^- is the schematic extension of NM logic obtained adding the axiom $\neg(\neg x^2)^2 \leftrightarrow (\neg(\neg x)^2)^2$.

On the algebraic side we have that an NM algebra is an NM^- algebra if and only if it does not have a negation fixpoint.

Since Definitions given for NM logic easily apply to the NM^- case, we can consider the idempotent Euler characteristic on free n-generated NM^- algebras.

Main result

Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \textsc{Form}_n$, the valuation $\chi^+(\varphi)$ equals the number of assignments $\mu\colon \textsc{Form}_n \to \{0,1\}$ such that $\mu(\varphi) = 1$.

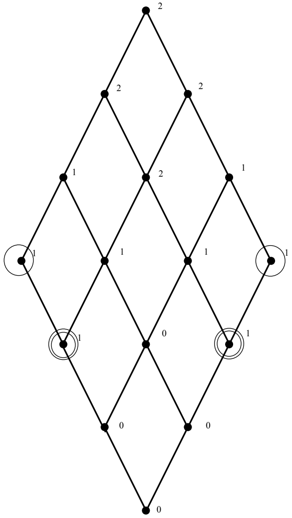
Main result

Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \textsc{Form}_n$, the valuation $\chi^+(\varphi)$ equals the number of assignments $\mu\colon \textsc{Form}_n \to \{0,1\}$ such that $\mu(\varphi) = 1$.

Remark. If φ is a tautology in NM^-, then $\chi^+(\varphi) = 2^n$.

Example



The Nilpotent Minimum algebra $\mathscr{NM}_1^-$, with the values of $\chi^+ \colon \mathscr{NM}_1^- \to \mathbb{R}$.

Further research

- Investigate a generalised idempotent Euler characteristic for NM logic, as done for Gödel logic.
- Investigate the logical content of the Euler characteristic in NM logic.

References

Euler Characteristic

- Klee, V.: The Euler characteristic in combinatorial geometry. Amer. Math. Monthly 70, 119–127 (1963)
- Rota, G.C.: On the combinatorics of the Euler characteristic. In: Studies in Pure Mathematics, 221–233. Academic Press, London (1971)

Gödel Logic

- Codara, P., D'Antona, O.M., Marra, V.: The Euler Characteristic of a Formula in Gödel Logic. 40th IEEE International Symposium on Multiple-Valued Logic (ISMVL) 2010, 108–112(2010)
- Codara, P., D'Antona, O.M., Marra, V.: Valuations in Gödel logic, and the Euler characteristic. Journal of Multiple-Valued Logic and Soft Computing 19(1-3), 71–84 (2012)

NM Logic

- Codara, P., Valota, D.: Valutations in Nilpotent Minimum Logic. 45th IEEE International Symposium on Multiple-Valued Logic (ISMVL) 2015, 90–95 (2015)

Thank you for your attention.