Indiscernibility Relations on Partially Ordered Sets

Pietro Codara

Dipartimento di Informatica e Comunicazione,
Università degli Studi di Milano

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Outline

1. Indiscernibility relation, the case of sets
2. Partitions of a partially ordered set
3. Indiscernibility relation, the case of posets
4. Further research

Indiscernibility relation

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$$(x, y) \in I_B \text{ if and only if } a(x) = a(y),$$

for each $a \in A$, and for each $x, y \in U$.

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"*The indiscernibility relation generated in this way is the mathematical basis of rough set theory.*" [Pawlak]

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In symbols, if $x, y \in U$, x is distinguishable from y if and only if $[x]_\pi \neq [y]_\pi$.

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To answer this question we need a definition of partition of a poset.

Partition of a poset, introduction

A partition of a set can be defined in terms of *fibres* of a surjection: a partition of a set A is the set $\{f^{-1}(y) \mid y \in B\}$ of fibres of a surjection $f : A \to B$.

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- P. Codara, "A theory of partitions of partially ordered sets," Ph.D. dissertation, Università degli Studi di Milano, Italy, 2008.
- ——, "Partitions of a Finite Partially Ordered Set," in *From Combinatorics to Philosophy. The Legacy of G.-C. Rota*, E. Damiani, O. D'Antona, V. Marra, and F. Palombi, Eds. New York: Springer US, 2009, pp. 45–59.
- P. Codara, O. M. D'Antona, and V. Marra, "Open Partitions and Probability Assignments in Gödel Logic," in *ECSQARU*, ser. LNCS (LNAI), vol. 5590. Springer, 2009, pp. 911–922.

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Such distinction leads us to introduce two different notions of partition of a poset, one based on the use of epimorphisms, the other based on the use of regular epimorphisms.

Blockwise preorder

Blockwise preorder

Let $(P, \leqslant)$ be a poset and let π be a partition of the set P. For $x, y \in P$, x *is blockwise under* y *with respect to* π, written $x \lesssim_\pi y$, if and only if there exists a sequence $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y \in P$ satisfying the following conditions.

(1) For all $i \in \{0, \ldots, n\}$, $[x_i] = [y_i]$.

(2) For all $i \in \{0, \ldots, n-1\}$, $y_i \leqslant x_{i+1}$.

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The definition of blockwise preorder allows us:

- to provide a characterization of regular epimorphisms;
- to provide intrinsic definitions of partitions of posets.

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Monotone partition

A *monotone partition* of a poset P is a poset $(\pi, \preceq)$ where

(i) π is a partition of the underlying set of P,

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Regular partition

A *regular partition* of a poset P is a poset $(\pi, \preceq)$ where

(i) π is a partition of the underlying set of P,

(ii) for each $x, y \in P$, $x \lesssim_\pi y$ if and only if $[x] \preceq [y]$.

Indiscernibility relation on a poset

The question:

> *Consider an information system $\mathcal{P} = (P, A)$ having as universe a finite poset $P = (U, \leqslant)$. Does an indiscernibility relation I_B on P induces a partition π on P?*

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Consider an information system $\mathcal{P} = (P, A)$ *having as universe a finite poset* $P = (U, \leqslant)$. *Does an indiscernibility relation* I_B *on* P *induces a partition* π *on* P*?*

The answer:

Not necessarily.

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Consider an information system $\mathcal{P} = (P, A)$ having as universe a finite poset $P = (U, \leqslant)$. Does an indiscernibility relation I_B on P induces a partition π on P?

The answer:

Not necessarily.

Compatibility

Let $P = (U, \leqslant)$ be a poset, let I_B be an indiscernibility relation on P, and let $\pi = U/I_B$. We say I_B is compatible with P if there exists a monotone partition $(\pi, \preceq)$ of P.

Compatibility criterion

Compatibility Criterion

Let $P = (U, \leqslant)$ be a poset, let I_B be an indiscernibility relation on P, and let $\pi = U/I_B$. Then, I_B is compatible with P if and only if, for all $x, y \in P$,

$$x \lesssim_\pi y \text{ and } y \lesssim_\pi x \text{ imply } [x]_\pi = [y]_\pi \,.$$

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Uniqueness

Let $P = (U, \leqslant)$ be a poset, let I_B be an indiscernibility relation on P, and let $\pi = U/I_B$. If I_B is compatible with P, then π admits a unique extension to a regular partition of P.

Example (1)

Consider the following table, reporting a collection of houses for sale in the city of Merate, Lecco, Italy.

House	Price (€)	Size (m^2)	District	Condition	Rooms
a	200.000	50	Centre	excellent	2
b	170.000	70	Centre	poor	3
c	185.000	53	Centre	very good	2
d	190.000	68	Sartirana	very good	3
e	140.000	60	Sartirana	good	2
f	155.000	65	Novate	good	2
g	250.000	85	Novate	excellent	3
h	240.000	75	Novate	excellent	3

Example (2)

Let $U = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h}\}$ be the set of all houses. We choose the subset of attributes $O = \{\text{Price}, \text{Size}\}$ to define on U a partial order $\leqslant$ as follow. For each $x, y \in U$,

$$x \leqslant y \text{ if and only if } \text{Price}(x) \leqslant \text{Price}(y)\,, \text{Size}(x) \leqslant \text{Size}(y)\,.$$

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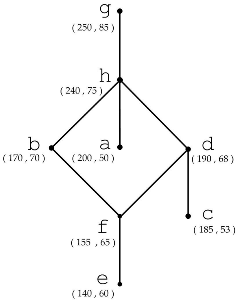
$$x \leqslant y \text{ if and only if } \text{Price}(x) \leqslant \text{Price}(y)\,, \text{Size}(x) \leqslant \text{Size}(y)\,.$$

We obtain the poset $P = (U, \leqslant)$.



P

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We denote by $\mathcal{P}(P, \bar{A})$ the information system having P as universe, and $\bar{A} = \{\text{District}, \text{Condition}, \text{Rooms}\}$ as the set of attributes.

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Let $D = \{\text{District}\}$, $C = \{\text{Condition}\}$, and $R = \{\text{Rooms}\}$, and denote by π_D, π_C, and π_R the partitions U/I_D, U/I_C, and U/I_R respectively. Moreover, let $S = D \cup R$, $T = C \cup R$, and let $\pi_S = U/I_S$, and $\pi_T = U/I_T$. We have:

$$\pi_D = \{\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}, \{\mathrm{d}, \mathrm{e}\}, \{\mathrm{f}, \mathrm{g}, \mathrm{h}\}\};$$

$$\pi_C = \{\{\mathrm{a}, \mathrm{g}, \mathrm{h}\}, \{\mathrm{b}\}, \{\mathrm{c}, \mathrm{d}\}, \{\mathrm{e}, \mathrm{f}\}\}\};$$

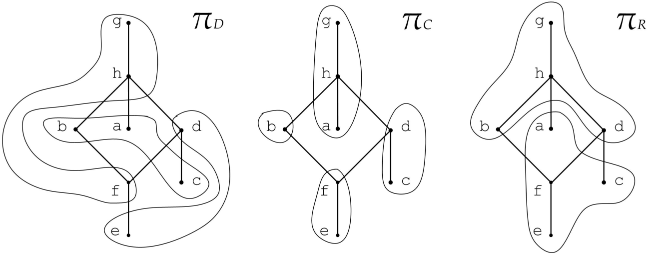
$$\pi_R = \{\{\mathrm{a}, \mathrm{c}, \mathrm{e}, \mathrm{f}\}, \{\mathrm{b}, \mathrm{d}, \mathrm{g}, \mathrm{h}\}\};$$

$$\pi_S = \{\{\mathrm{a}, \mathrm{c}\}, \{\mathrm{b}\}, \{\mathrm{d}\}, \{\mathrm{e}\}, \{\mathrm{f}\}, \{\mathrm{g}, \mathrm{h}\}\};$$

$$\pi_T = \{\{\mathrm{a}\}, \{\mathrm{b}\}, \{\mathrm{c}\}, \{\mathrm{d}\}, \{\mathrm{e}, \mathrm{f}\}, \{\mathrm{g}, \mathrm{h}\}\}.$$

Example (4)

The following figure shows the partitions π_D, π_C, and π_R.



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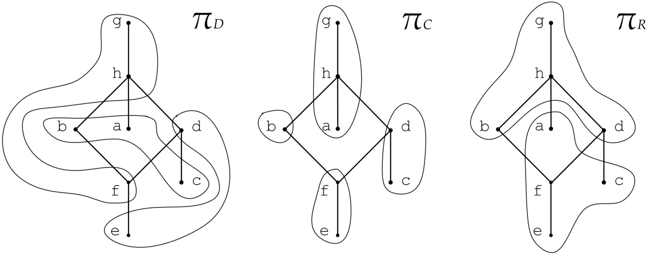
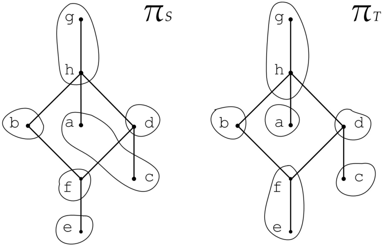


Using the compatibility criterion one can check that while π_C and π_R are compatible with P, π_D is not.

Example (5)

The following figure shows the partitions π_S, and π_T.



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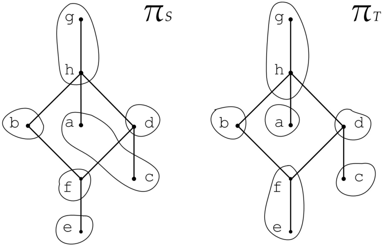

Using the compatibility criterion one can check that π_S and π_T are compatible with P, and thus can be extended to monotone partitions of P.

Further research

- Investigation of the structure of all possible indiscernibility relations on a partially ordered universe.
- In some sense, when we consider the (unique) regular partition of a poset P induced by an indiscernibility relation compatible with P, we are not adding information to the partial order of P. On the other hand, taking into account monotone partitions which are not regular, amounts to add information which was not previously included in the poset. *Can this be regarded as an attempt to add some preferences amongst the elements of the poset?*
- ...

Thank you for your attention.