A Characterisation of Bases of Triangular Fuzzy Sets

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Motivation and Aim

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- In our main result we prove that a reasonable set of such conditions suffices to characterise families of triangular fuzzy sets.
- A second result provides an additional characterisation of such families in terms of properties of the curve that they parametrise.

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Ruspini partitions

By a *fuzzy set* we mean a function $f\colon [0,1] \to [0,1]$. Throughout this presentation, we fix a finite nonempty family of fuzzy sets $P = \{f_1, \ldots, f_n\}$.

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Definition

P is a *Ruspini partition* if for all $x \in [0,1]$

$$\sum_{i=1}^{n} f_i(x) = 1 .$$

Overlapping

Definition

We say P is 2-*overlapping* if for all $x \in [0,1]$ and all triples of indices $i_1 \neq i_2 \neq i_3$ one has

$$\min\{f_{i_1}(x), f_{i_2}(x), f_{i_3}(x)\} = 0 \; .$$

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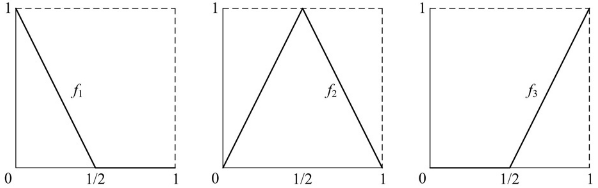

Figure: A 2-overlapping Ruspini partition $\{f_1, f_2, f_3\}$.

Normality

The Ruspini and the 2-overlapping conditions apply to a family of fuzzy sets. Other properties that we consider apply to a single fuzzy set.

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Definition

A fuzzy set $f\colon [0,1] \to [0,1]$ is *normal* if there exist $x \in [0,1]$ such that $f(x) = 1$.

If, moreover, $f(y) \neq 1$ for all $y \in [0,1]$ with $y \neq x$, we say that f is *strongly normal*.

Convexity

Classically, $f\colon [0,1] \to [0,1]$ is *convex* if for all $x, y, \lambda \in [0,1]$, with $x \neq y$,

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y). \tag{1}$$

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Definition

The function f is *min-convex* if for all $x, y, \lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \geq \min(f(x), f(y)),$$

and it is *strictly min-convex* if for $\lambda \in (0,1)$

$$f(\lambda x + (1-\lambda)y) > \min(f(x), f(y)).$$

min-convexity, an example

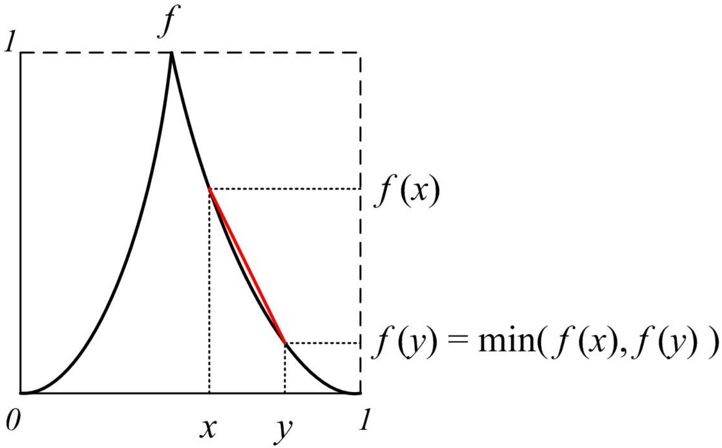

Figure: A min-convex function which is not convex.

Local Convexity

Let us call $S_f = \{x \in [0,1] \mid f(x) > 0\}$ the *support* of f.

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We say f is *convex on its support* if

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holds for each $x, y \in [0,1]$ such that $[x, y] \subseteq S_f$.

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We define the notions of (*strict*) *min-convexity of f on its support* in the same manner, *mutatis mutandis*.

Triangular basis of fuzzy sets

Definition

A finite family $P = \{f_1, \dots, f_n\}$ of continuous fuzzy sets is a *pseudo-triangular basis* if there exist $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ such that (up to a permutation of the indices) for each $i = 1, \dots, n-1$

- *a*) $f_i(t_i) = 1$, $f_i(t_{i+1}) = 0$,
- *b*) $f_j(x) = 0$, for $x \in [t_i, t_{i+1}]$, $j \neq i, i+1$,
- *c*) $f_{i+1}(x) = 1 - f_i(x)$, for $x \in [t_i, t_{i+1}]$, and
- *d*) f_i, f_{i+1} are bijective when restricted to $[t_i, t_{i+1}]$.

Further, P is a *triangular basis* if the following condition holds in place of *d*).

d*) f_i, f_{i+1} are linear over $[t_i, t_{i+1}]$.

Triangular basis of fuzzy sets, examples

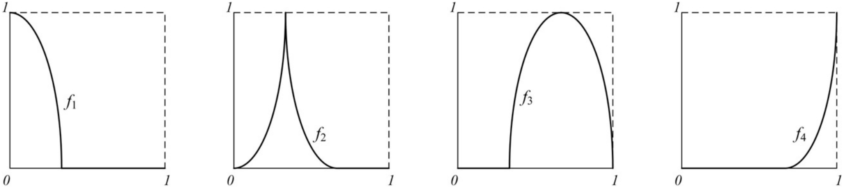

Figure: A pseudo-triangular basis of fuzzy sets.

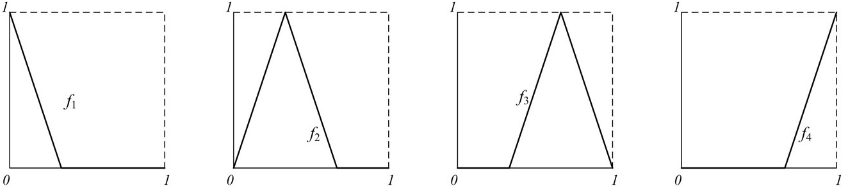

Figure: A triangular basis of fuzzy sets.

Pseudo-triangular basis and general properties of fuzzy sets

Lemma

The following are equivalent.

- ***i*)** P is a 2-overlapping Ruspini partition and each f_i is strongly normal, min-convex, and strictly min-convex on its support.
- ***ii*)** P is a pseudo-triangular basis.

Proof of the Lemma

- Strong normality + Ruspini
  a) $f_i(t_i) = 1$, $f_i(t_{i+1}) = 0$, for $0 \leq t_1 < t_2 < \cdots < t_{n-1} < t_n \leq 1$

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- $\cdots$ + 2-overlapping + min-convexity
  — $t_1 = 0$, $t_n = 1$
  b) $f_j(x) = 0$, for $x \in [t_i, t_{i+1}]$, $j \neq i, i+1$,
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- $\cdots$ + strict min-convexity on the supports (+ continuity)
  - **d)** f_i, f_{i+1} are bijective when restricted to $[t_i, t_{i+1}]$

Triangular basis and general properties of fuzzy sets

Theorem

The following are equivalent.

- ***i*)** P is a 2-overlapping Ruspini partition, and each f_i is strongly normal, min-convex, and convex on its support.
- ***ii*)** P is a triangular basis.

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Hamiltonian path

The *fundamental simplex* in $\mathbb{R}^n$, denoted by Δ_n, is the convex hull of the standard basis of $\mathbb{R}^n$; the latter is denoted $\{e_1, \ldots, e_n\}$. In symbols, $\Delta_n = \mathrm{Conv}\,\{e_1, \ldots, e_n\}$.

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We say Θ is a *Hamiltonian path* if there is a permutation $\pi : \underline{n} \to \underline{n}$ such that

$$\Theta = \bigcup_{i=1}^{n-1} \mathrm{Conv}\{e_{\pi(i)}, e_{\pi(i+1)}\}$$

Pseudo-triangular basis and curve that the fuzzy sets parametrise

We define a continuous map $T : [0,1] \to [0,1]^n$ associated with P by

$$t \mapsto (f_1(t), \ldots, f_n(t)) \, .$$

We write $\Theta = T([0,1])$ for the range of T.

Corollary

The following are equivalent.

i) P is a 2-overlapping Ruspini partition, and each f_i is strongly normal, min-convex, and strictly min-convex on its support.

ii) The map $T : [0,1] \to [0,1]^n$ is injective, and Θ is a Hamiltonian path on $\Delta_n^{(1)}$.

Range parametrised by a triangular basis, examples

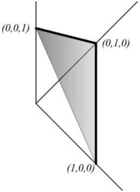


Figure: Range parametrised by a (pseudo-)triangular basis with 3 functions.

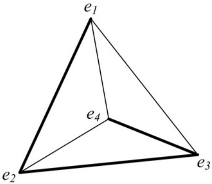


Figure: Range parametrised by a (pseudo-)triangular basis with 4 functions.

Range parametrised by a non-Ruspini family, example

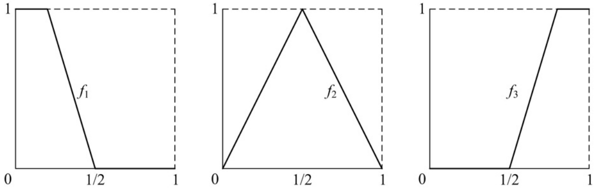
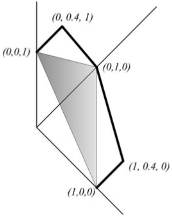


Figure: A non-Ruspini family $\{f_1, f_2, f_3\}$.



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- $\cdots$ + continuity
  - $\bigcup_{i=1}^{n-1} \mathrm{Conv}\{e_{\pi(i)}, e_{\pi(i+1)}\} \subseteq \Theta$

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- $\cdots$ + strict min-convexity on the supports
  - T is injective

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- In this paper, we focused on fuzzy sets whose domain is the real unit interval $[0, 1]$.
- Sometimes it may be necessary to deal with functions defined over the real unit n-cube $[0, 1]^n$.
- A natural question is whether our Theorem admits a generalisation to higher dimensions (triangular bases over $[0, 1]^n$).

Thanks

- Thank you for your attention.