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Propositional Gödel Logic and Delannoy Paths

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Propositional Gödel Logic

Gödel logic can be semantically defined as the logic of the minimum triangular norm.

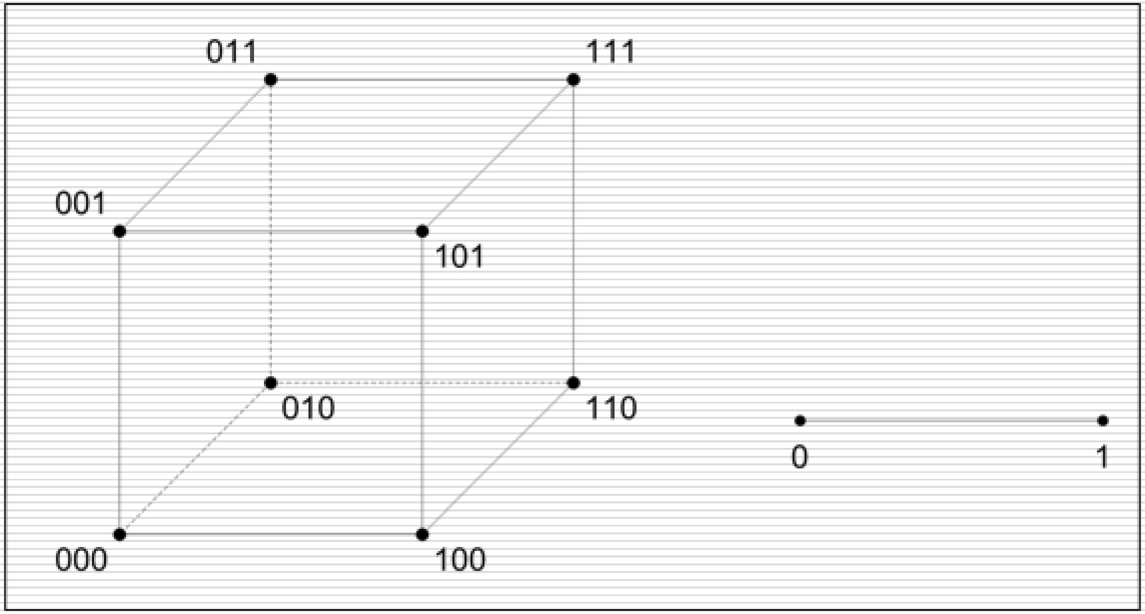
$$x \wedge y = \min(x, y) \quad x \to y = \begin{cases} 1 & \text{if } x \leq y\,, \\ y & \text{otherwise}\,. \end{cases}$$

$$x \vee y = \max(x, y) \quad \neg x = \begin{cases} 1 & \text{if } x = 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

Boolean Functions

The algebraic counterpart of the classical propositional logic can be given in terms of Boolean functions, with $\wedge$, $\vee$, $\rightarrow$, and $\neg$ defined pointwise.

$$f : \{0,1\}^n \rightarrow \{0,1\}$$



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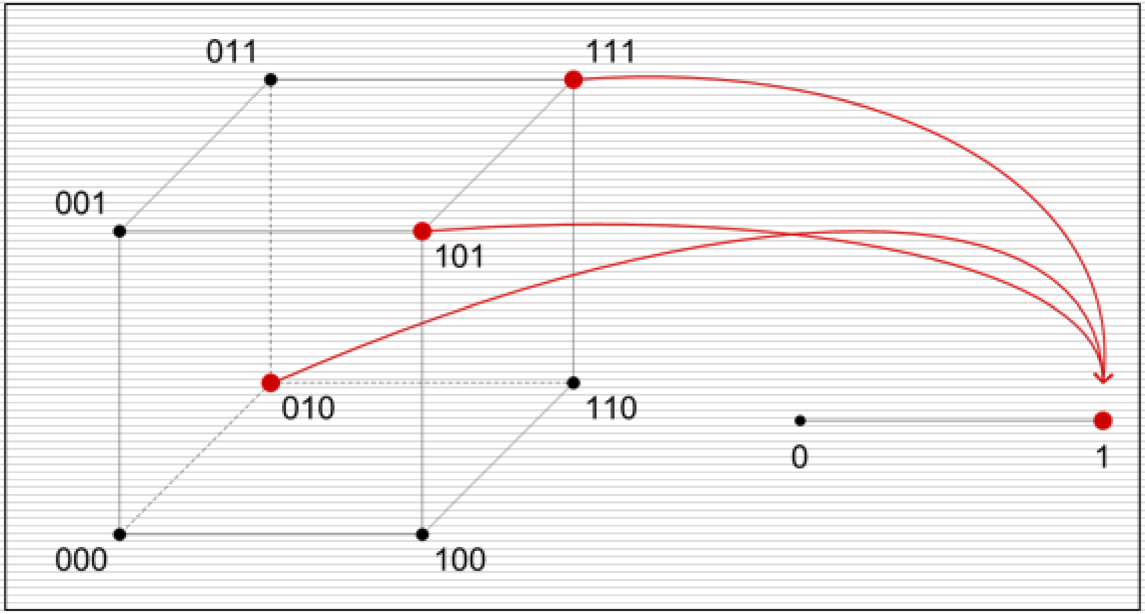

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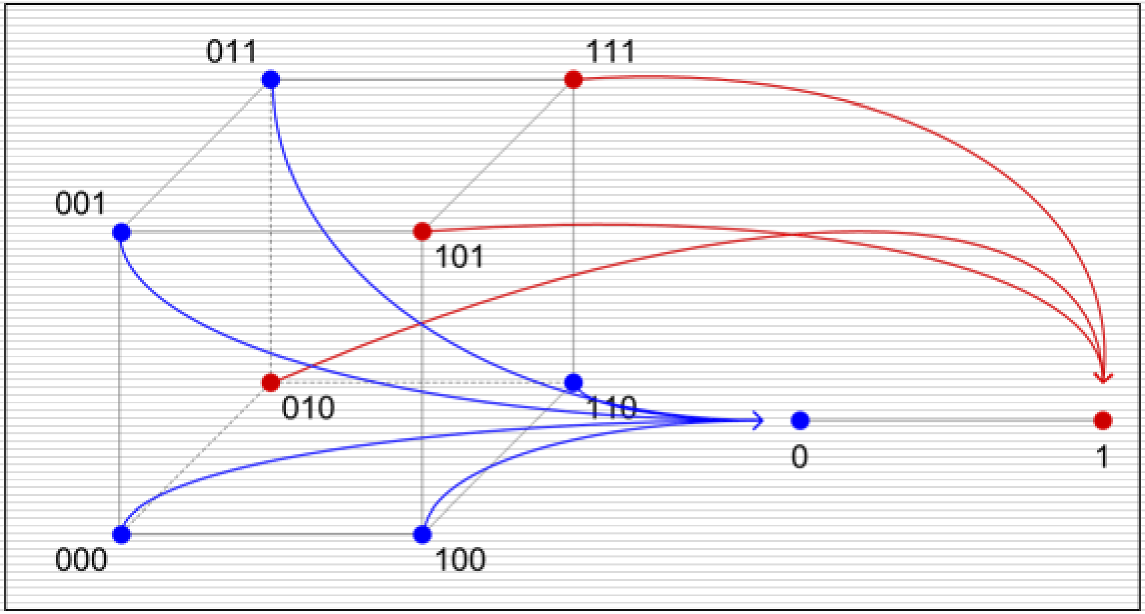

Enrichment of the n-cube

- How can we “enrich” the Boolean n-cube in order to represent the algebraic counterpart of Gödel logic (the free Gödel algebra) ?

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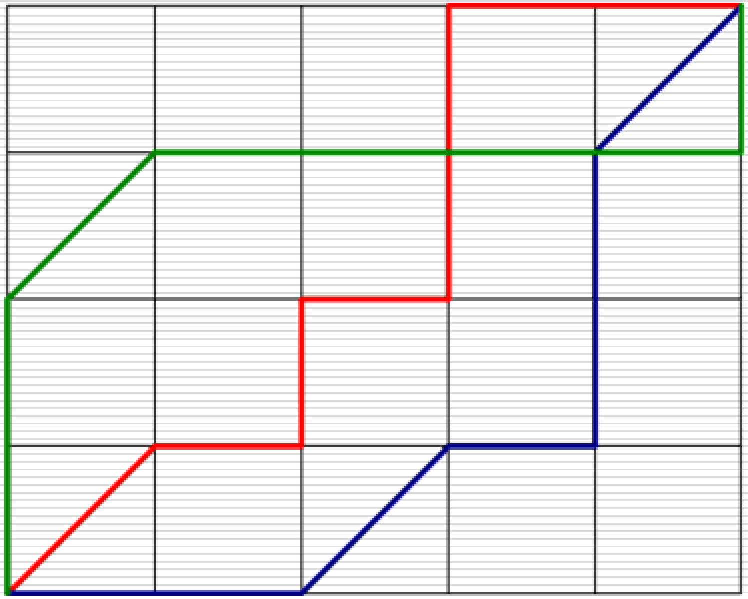
- The answer is given by Delannoy paths.

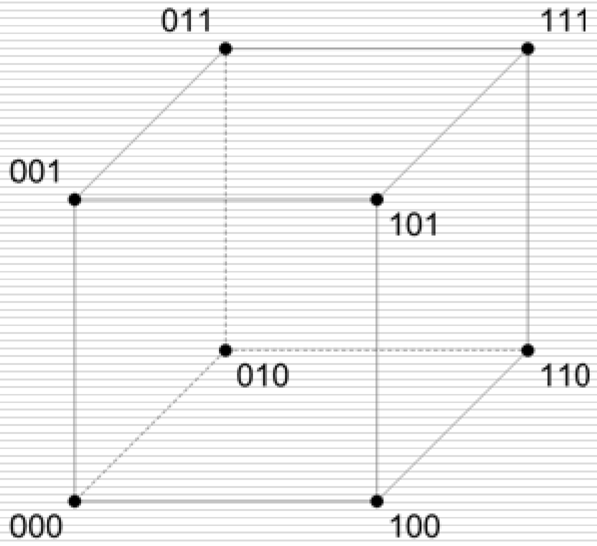
Delannoy Paths



Henri-Auguste Delannoy
(1833-1915)
French mathematician

- A Delannoy path is a path in $\mathbb{Z}^2$ that only uses northward, eastward and northeastward steps.

From Boolean to Delannoy n-cube



- The Boolean 3-cube.

- We enrich the Boolean n-cube with a variant of Delannoy paths.
- We indicate with $\mathcal{P}_n$ the n-cube so enriched.

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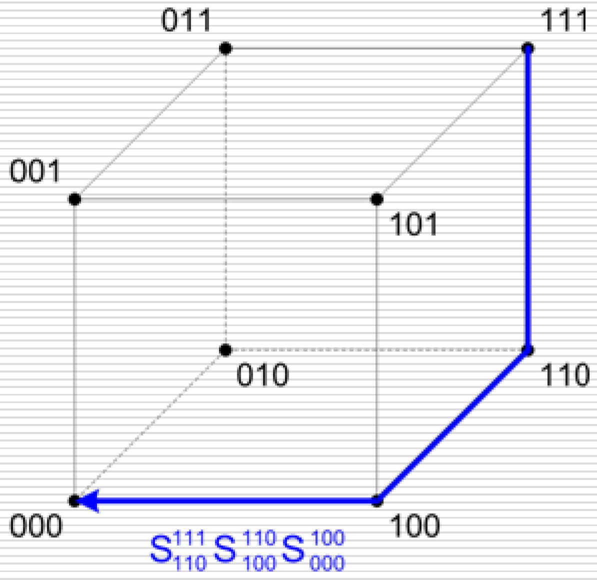

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➢A Delannoy path from 111 to 000, in the 3-cube.

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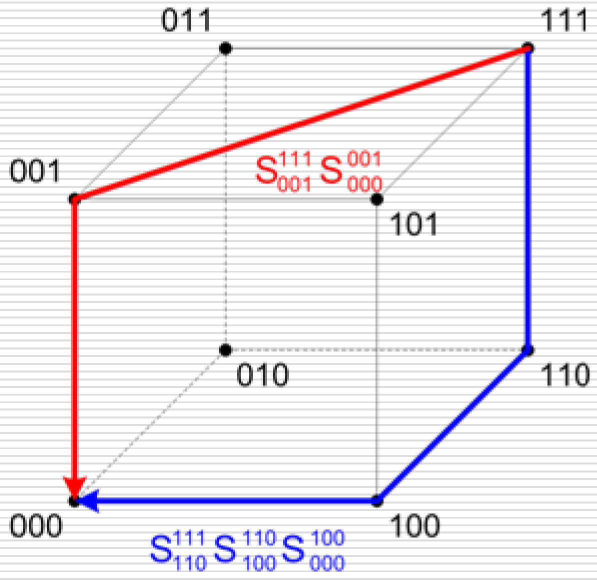

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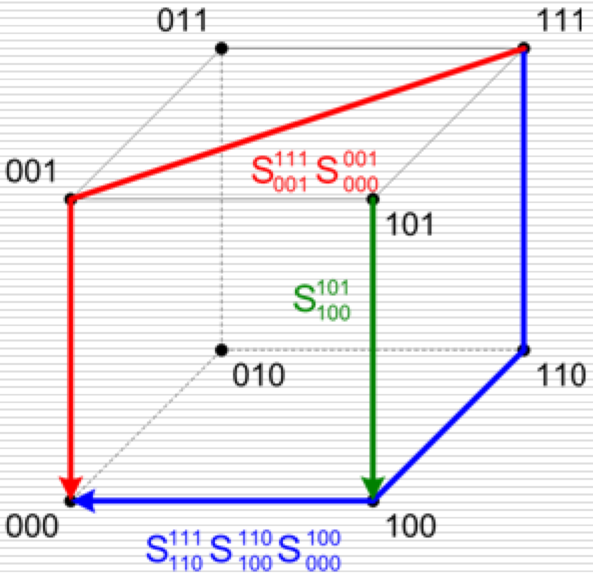

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Another Delannoy path in the 3-cube, with root 111.

➢ A Delannoy path with root 101, ending in 100.

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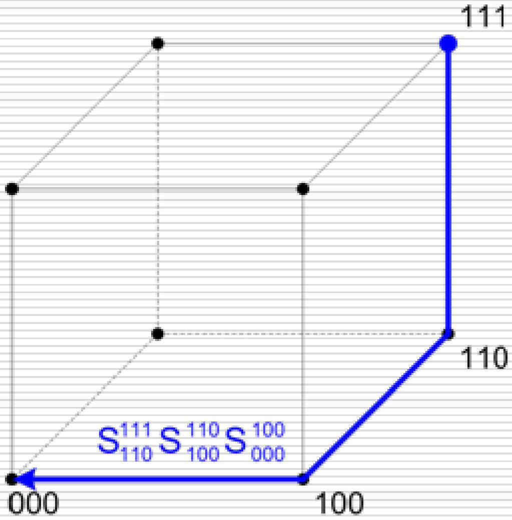
Delannoy Paths and Subpaths



➢A Delannoy path P.

- A subpath of a Delannoy path P is a path having the same root as P, consisting of some initial, consecutive, steps of P (in other words, it is a prefix of P.)
- $\Downarrow P$ is the family of all subpaths of P.

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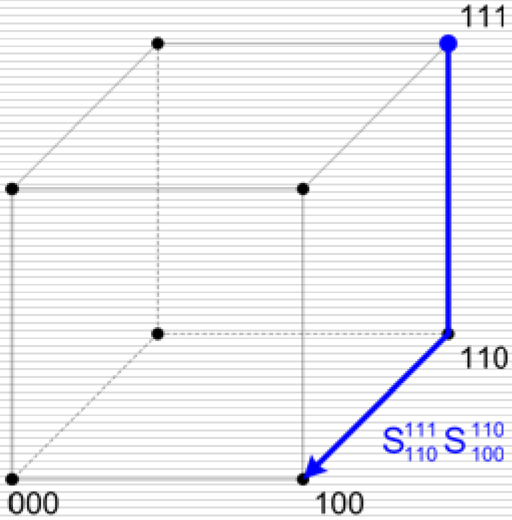

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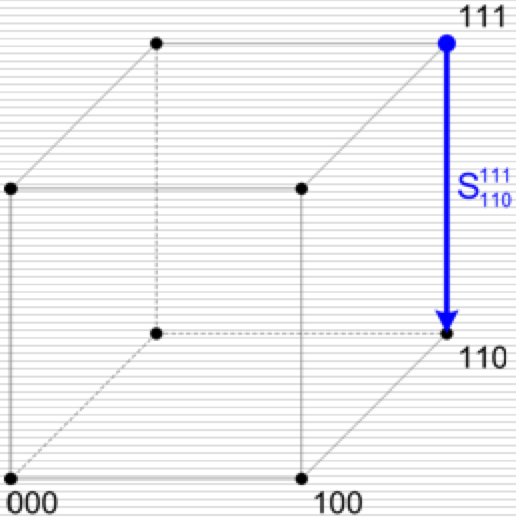

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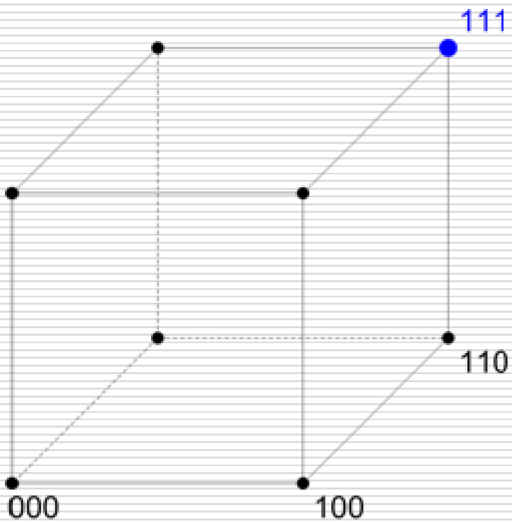

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Another subpath of P.

- The shortest subpath of P: its root 111.

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From Boolean Functions to Delannoy Functions

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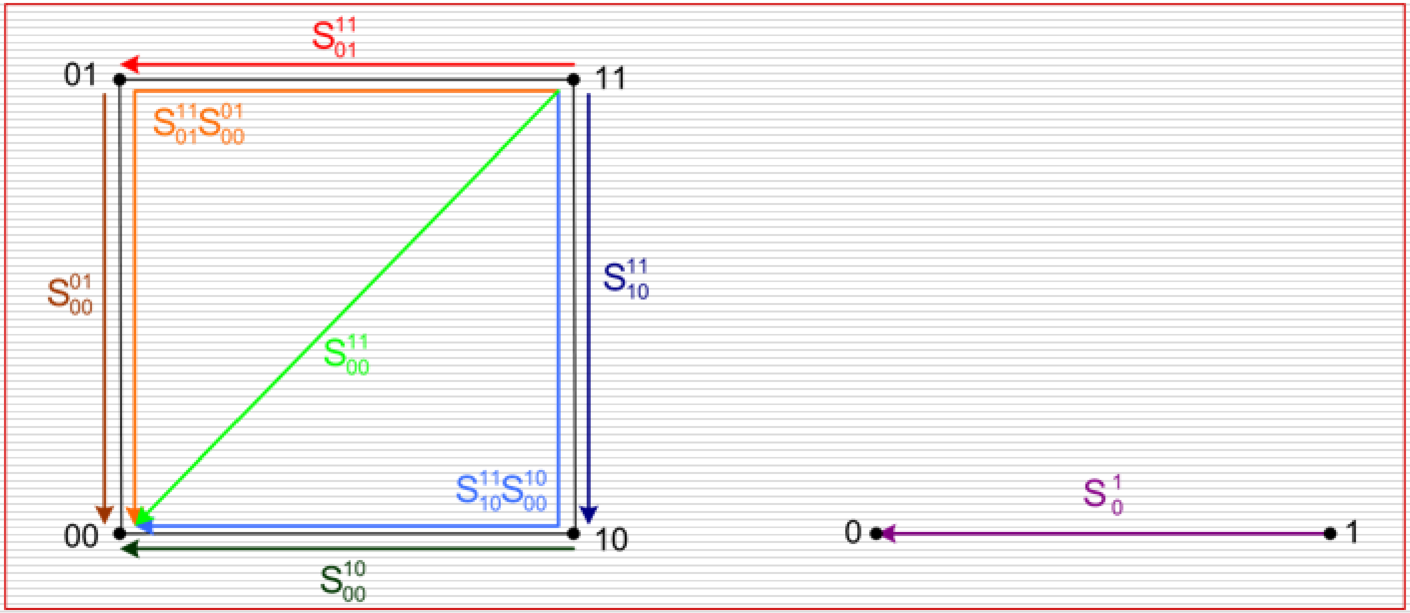
$$f : \{0,1\}^n \to \{0,1\}$$

- Delannoy Functions:

$f : \mathcal{P}_n \to \mathcal{P}_1$, with $f(\Downarrow P) = \Downarrow f(P)$.

Delannoy Functions

- We denote by $\mathcal{D}_n$ the set of all Delannoy functions from $\mathcal{P}_n$ to $\mathcal{P}_1$.



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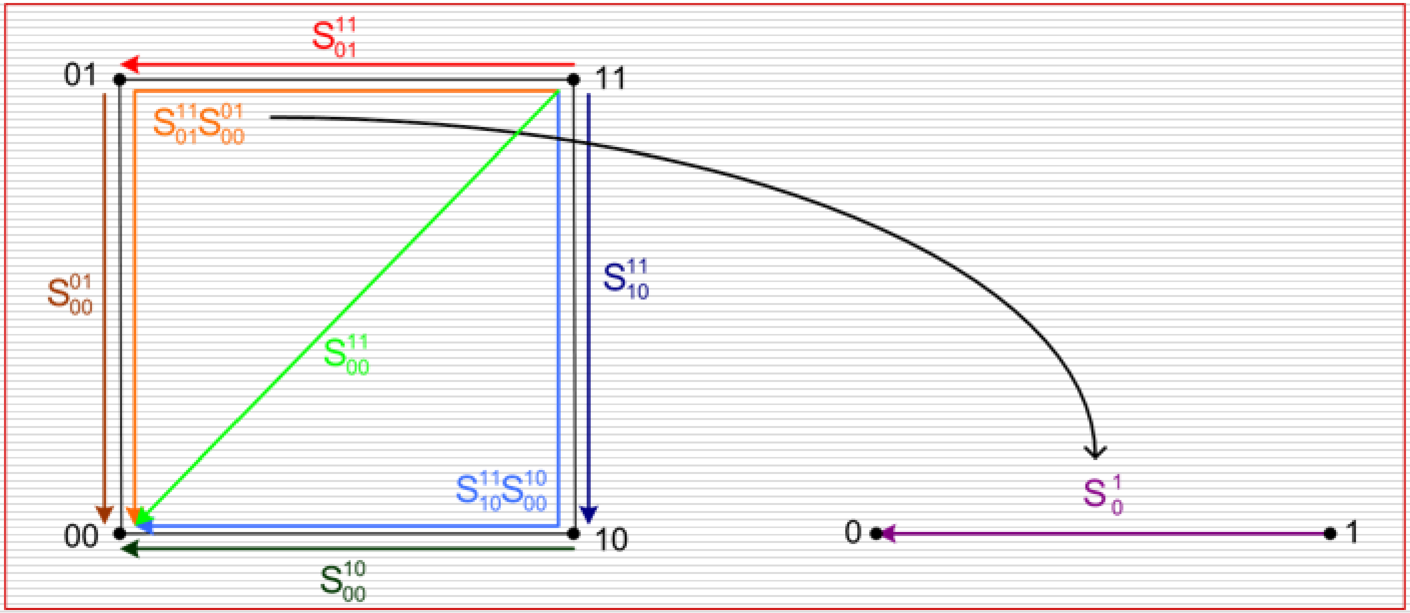

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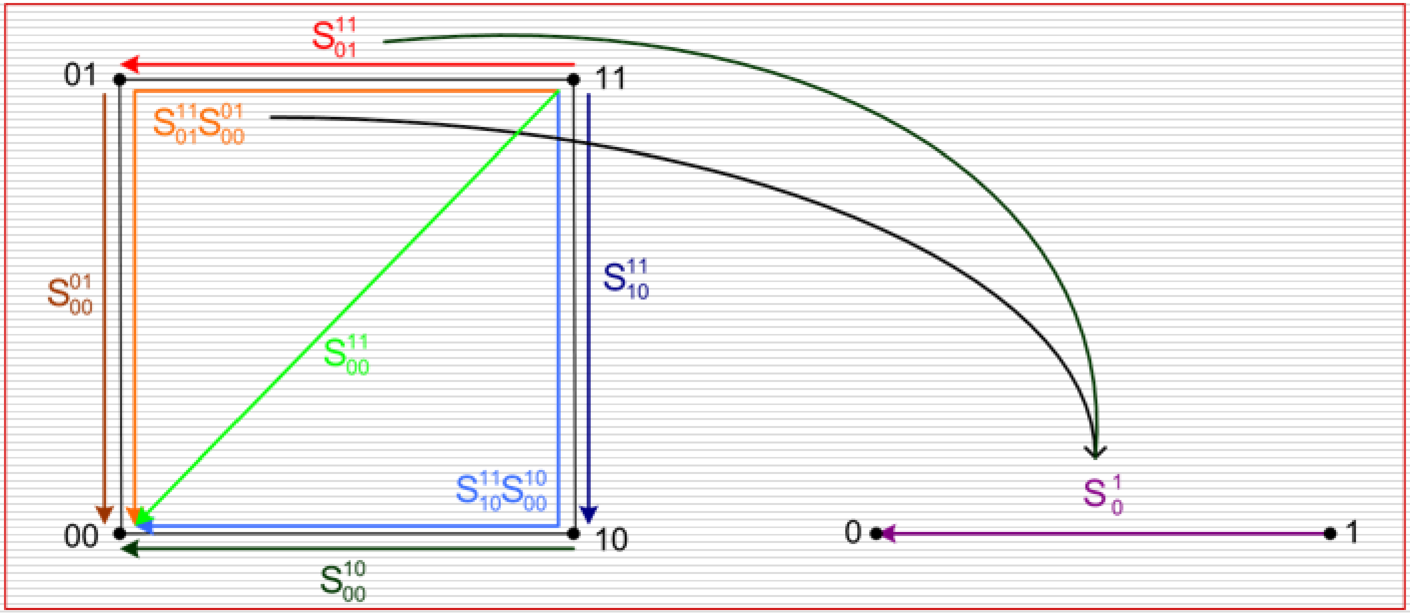

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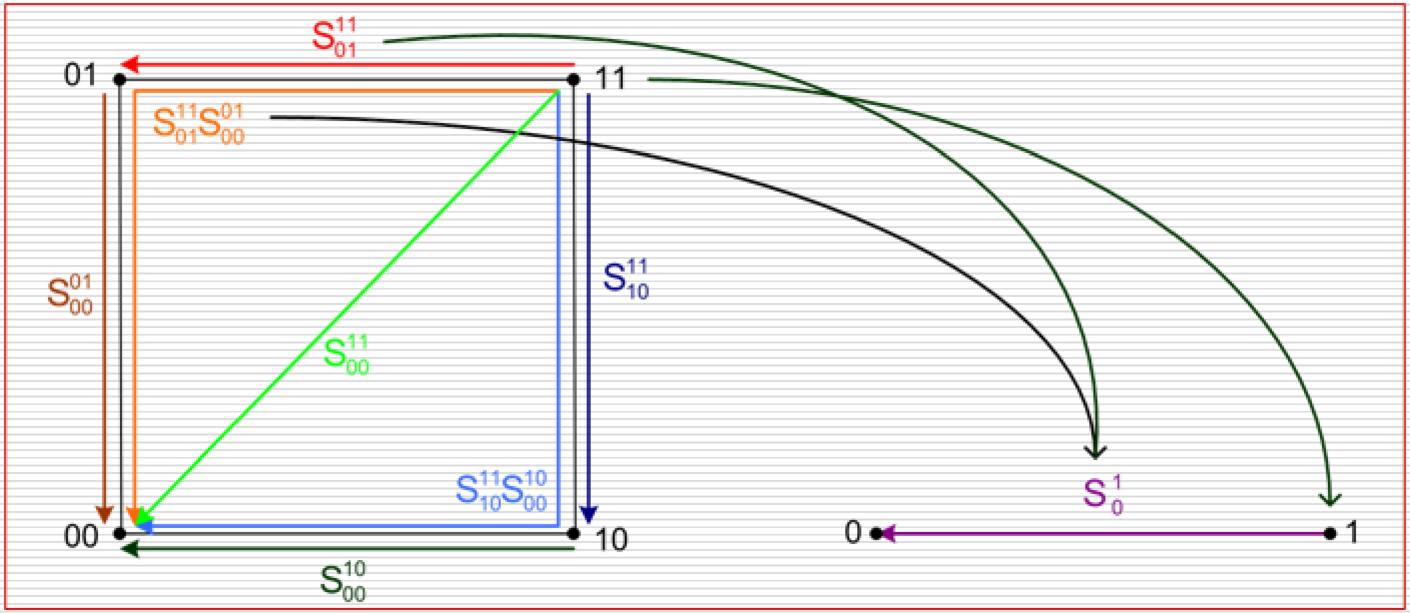

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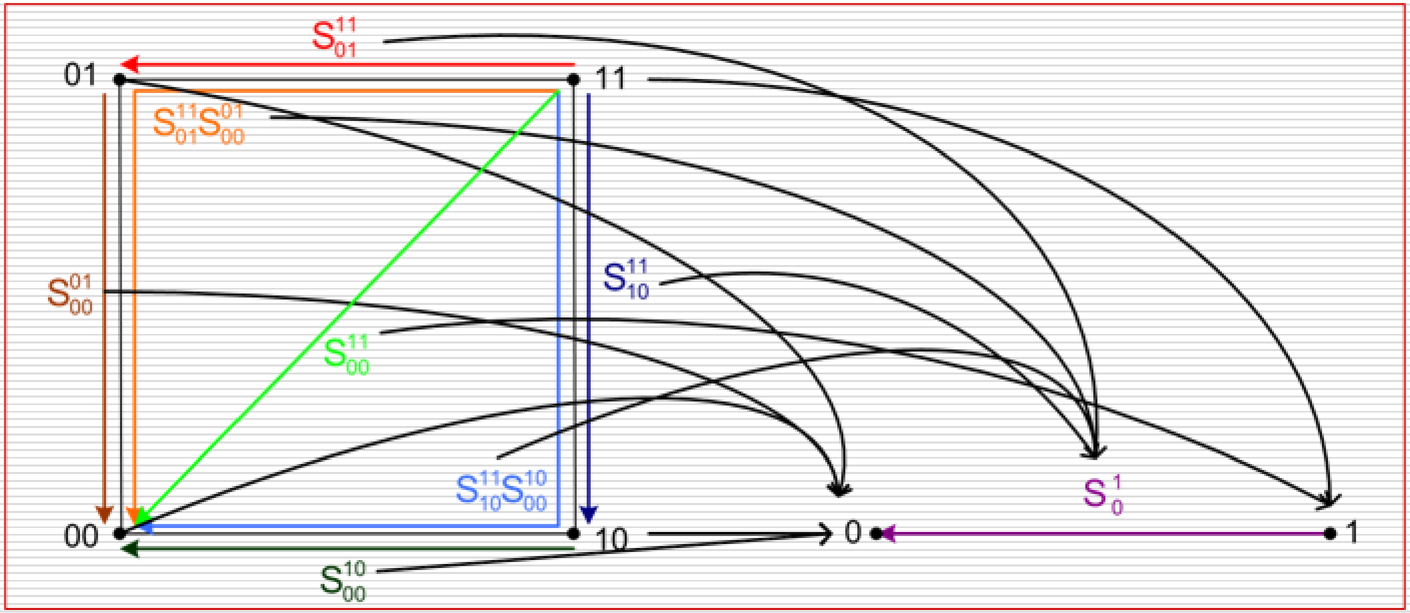

The Algebra of Delannoy Functions

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- $\wedge$ and $\vee$ induce a lattice order on $\mathcal{D}_n$

$$f \leqslant g \iff f^{-1}(1) \subseteq g^{-1}(1)$$

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$$f \leqslant g \iff f^{-1}(1) \subseteq g^{-1}(1)$$

- This works because a Delannoy function f is uniquely determined by its 1-set $f^{-1}(1)$, just like in the Boolean case.

Main Result

Theorem. $\langle \mathcal{D}_n , \wedge , \vee , \to , \neg \rangle$ is the free n-generated Gödel algebra.

Constructing the Free 1-generated Gödel Algebra

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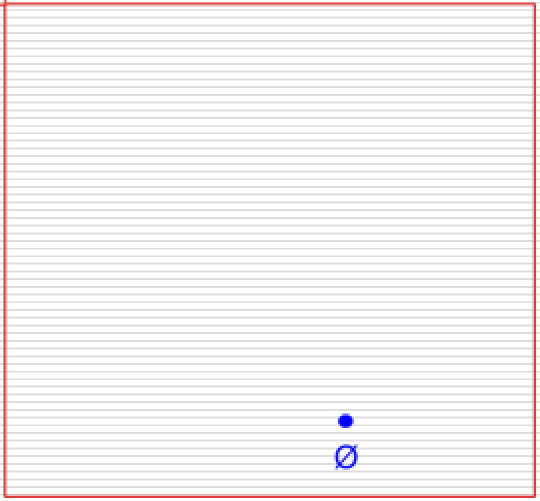
1-set:

$$f^{-1}(1) = \emptyset$$



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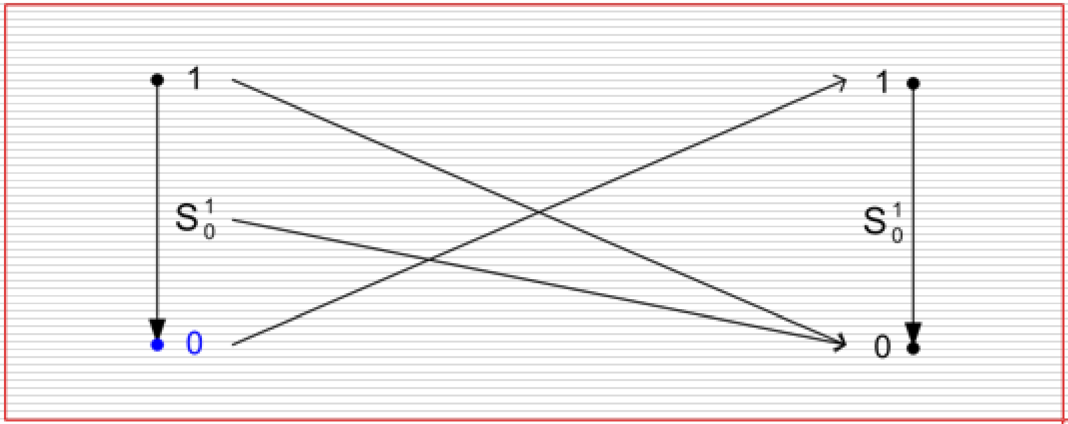
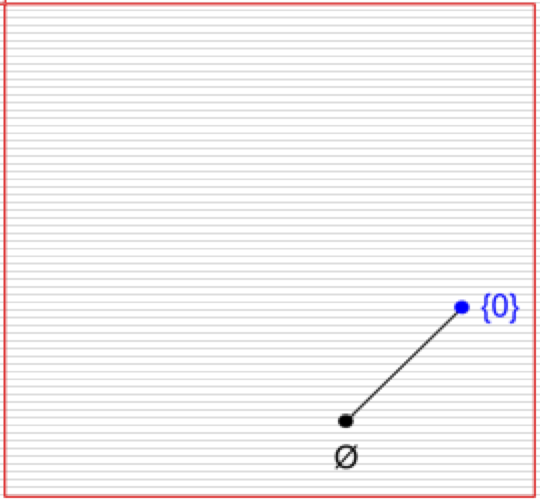


1-set:

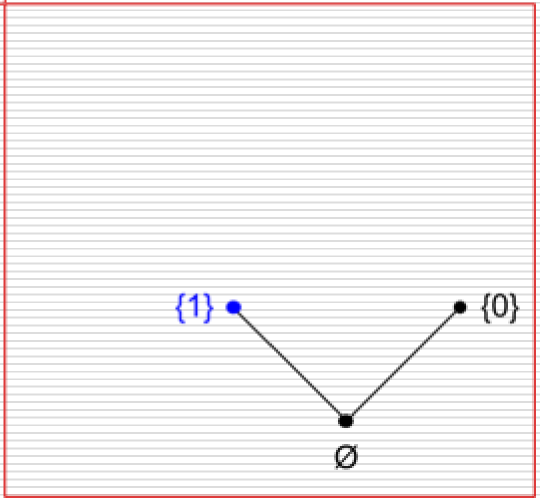
$$f^{-1}(1) = \{0\}$$



Constructing the Free 1-generated Gödel Algebra

1-set:

$$f^{-1}(1) = \{1\}$$



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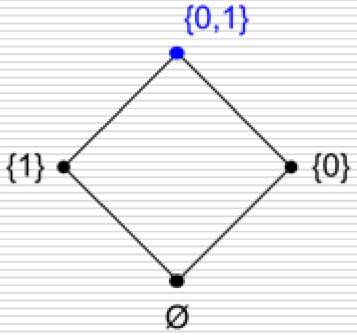

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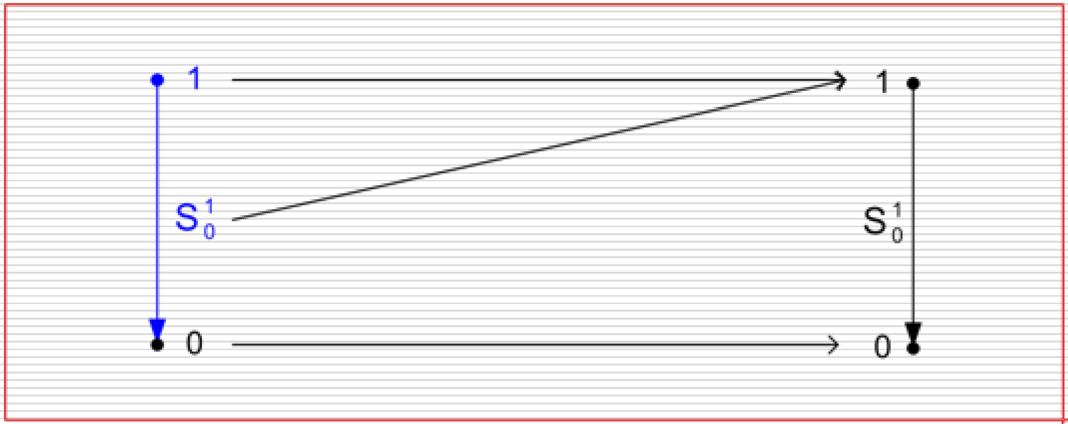
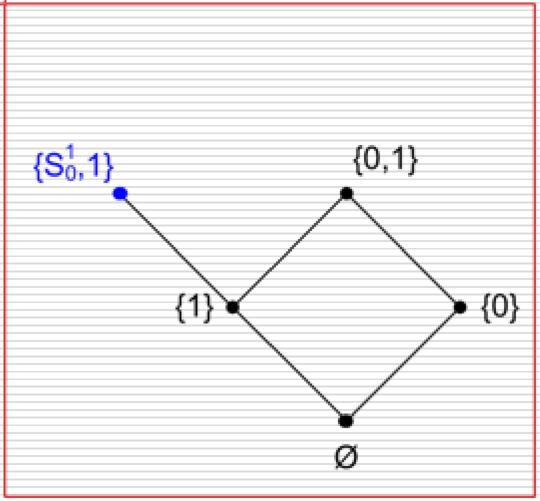


1-set:

$$f^{-1}(1) = \{S_0^1, 1\}$$



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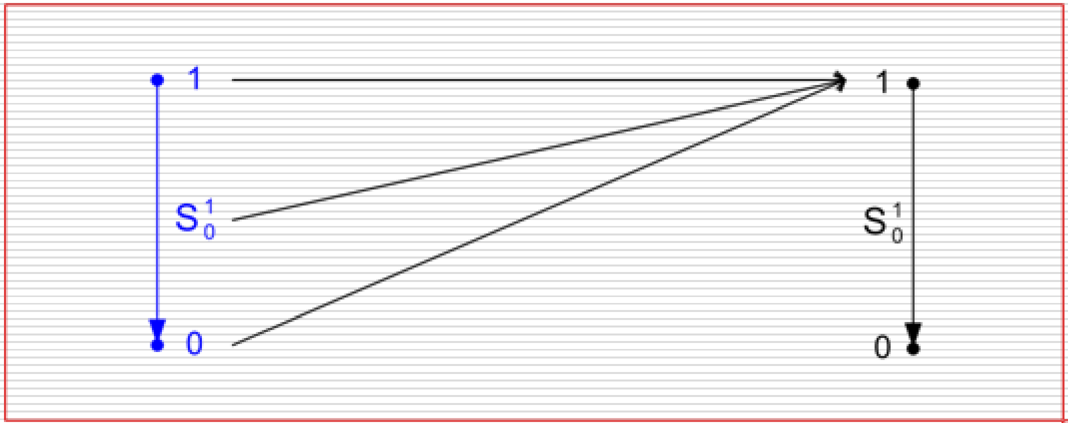
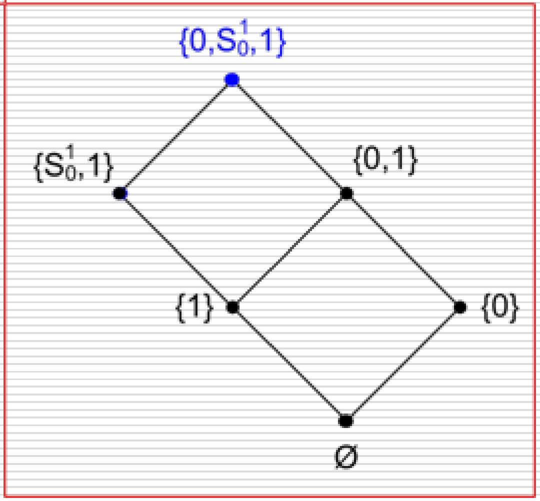


1-set:

$$f^{-1}(1) = \{0, S_0^1, 1\}$$



Future Work

- D'Antona and Marra, *Computing coproducts of finitely presented Gödel algebras*, Ann. Pure Appl. Logic, 2006.
- Aguzzoli, Gerla, and Marra, *Gödel algebras free over finite distributive lattices*, preprint. Generalized Delannoy Paths over posets (and not just cubes).

Grazie per l’attenzione

thank you

gracias

merci

gràcies

danke

obrigado

dank u

спасибо

tack

tak

teşekkürler