Best Approximation of Ruspini Partitions in Gödel Logic

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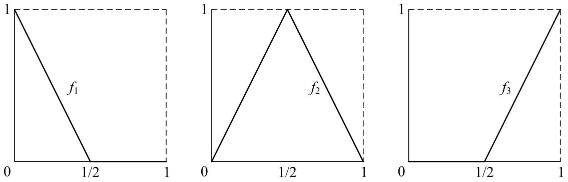
- Throughout this presentation, we fix a finite nonempty family of fuzzy sets $P = \{f_1, \ldots, f_n\}$.
- In our paper we deal with particular families of fuzzy sets: Ruspini partitions.

Definition

P is a *Ruspini partition* if for all $x \in [0,1]$

$$\sum_{i=1}^{n} f_i(x) = 1\,.$$

Motivation and Aim



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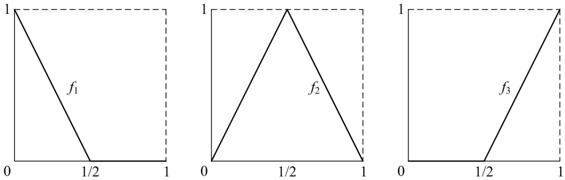

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- $X_1 =$ “The temperature is low”.

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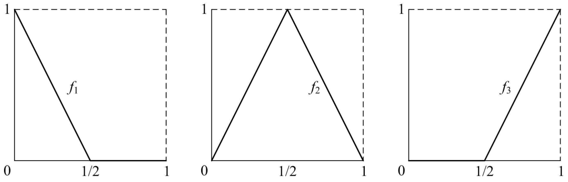

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- $X_2 =$ “The temperature is medium”.

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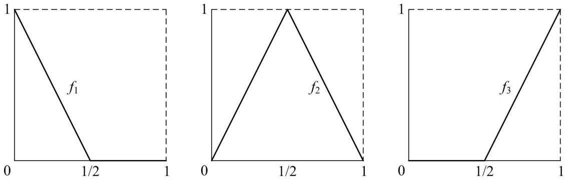

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- $X_1 =$ “The temperature is low”.
- $X_2 =$ “The temperature is medium”.
- $X_3 =$ “The temperature is high”.

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- The set P leads one to add extra-logical axioms to $\mathscr{L}$, *e.g.* $\neg(X_1 \wedge X_3)$, in an attempt to express the fact that one cannot observe both a high and a low temperature at the same time. More generally, P implicitly encodes a *theory* over the pure logic $\mathscr{L}$.

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- Our work provides an analysis of how the Ruspini condition on P is reflected by the resulting theory over $\mathscr{L}$.
- We take $\mathscr{L}$ to be Gödel logic.

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  $\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$
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  $\mu(\alpha \rightarrow \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases}$
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- A *tautology* is a formula α such that $\mu(\alpha) = 1$ for every assignment μ.
- Gödel logic is complete with respect to this many-valued semantics.

The Theory of P

- Suppose $f_1, \ldots, f_n$ provide truth values for the propositions $X_1, \ldots, X_n$ in Gödel logic. We can describe the theory encoded by $P = \{f_1, \ldots, f_n\}$ by the set of axioms $\Theta(P) = \{\alpha(X_1, \ldots, X_n) \,|\, \mu(\alpha) = 1 \;\forall \mu \text{ s.t. } \exists x \;\forall i \;\mu(X_i) = f_i(x)\}$

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- In Gödel logic $\Theta(P)$ is finitely axiomatizable, because the number of variables is finite. Thus, there exists a single axiom α_P which axiomatizes $\Theta(P)$, that is $\forall \beta \in \textsc{Form}\,,\ \alpha_P \vdash \beta \Leftrightarrow \beta \in \Theta(P)\,.$

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- We note that α_P is uniquely determined by P up to logical equivalence.
- α_P encodes all relations between the fuzzy sets $f_1, \ldots, f_n$ that Gödel logic is capable to express.

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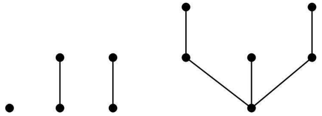
- The answer is no.
- However, we prove that Gödel logic does capture a weaker Ruspini condition.
- We will show that our weaker Ruspini condition indeed is the best approximation of Ruspini partitions in Gödel logic.

Forests and Subforests

- We need to introduce a specific forest built from assignments that plays a key role in the following.
- Recall that, given a poset $(F, \leq)$ and a set $Q \subseteq F$, the *downset* of Q is

$$\downarrow Q = \{x \in F \mid x \leq q, \text{ for some } q \in Q\}.$$

- A poset F is a *forest* if for all $q \in F$ the downset $\downarrow \{q\}$ is a chain (*i.e.*, a totally ordered set). A *subforest* of a forest F is the downset of some $Q \subseteq F$.



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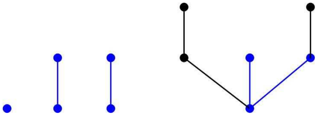

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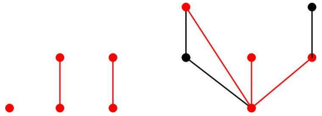

This is not a subforest.

The Forest $\mathscr{F}_n$

- We say that two assignments μ and ν are *equivalent over the first n-variables*, written $\mu \equiv_n \nu$, if and only if for any well-formed formula $\alpha(X_1, \ldots, X_n)$ in Gödel logic

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- We define an appropriate partial order $\leq$ on $\mathscr{F}_n$.
- $(\mathscr{F}_n, \leq)$ is a forest.

Examples: $\mathscr{F}_1$ and $\mathscr{F}_2$

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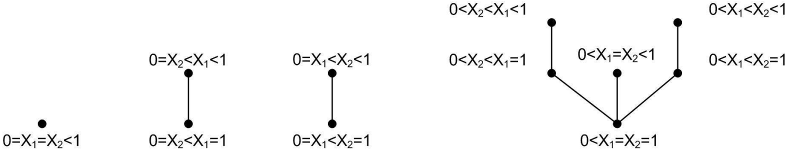

The forest $\mathscr{F}_2$.

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- For each subforest $F \subseteq \mathscr{F}_n$, there exists a formula $\alpha(X_1, \ldots, X_n)$ such that $\mathscr{F}_\alpha = F$.

- We associate with the family of the fuzzy sets P the uniquely determined subforest $\mathscr{F}_{\alpha_P} \subseteq \mathscr{F}_n$.

The Forest $\mathscr{F}(P)$

- Let $[\mu]_{\equiv_n} \in \mathscr{F}_n$ and $x \in [0,1]$. We say $[\mu]_{\equiv_n}$ is *realized by P at x* if there exists a permutation $\sigma : \underline{n} \to \underline{n}$ such that

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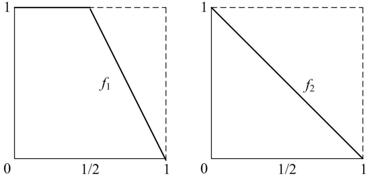
Examples: The Forest $\mathscr{F}(P)$



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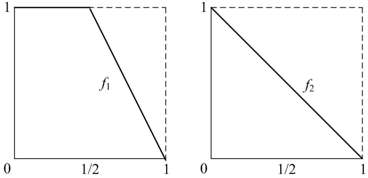

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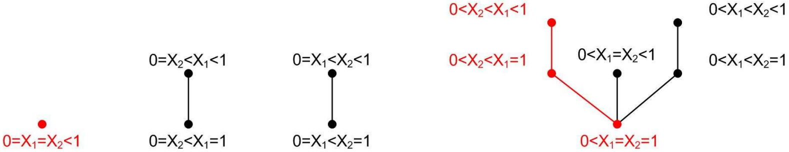

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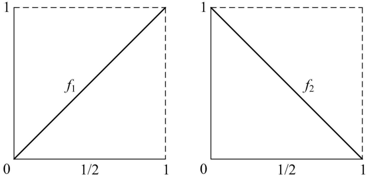

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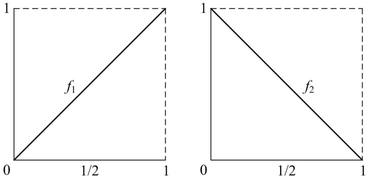
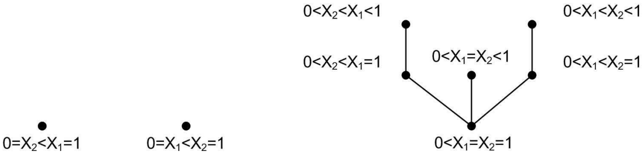


Then, $\mathscr{F}(P)$ is the following.



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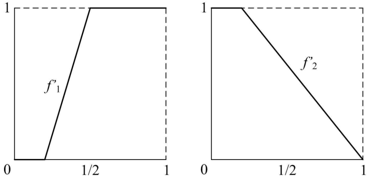

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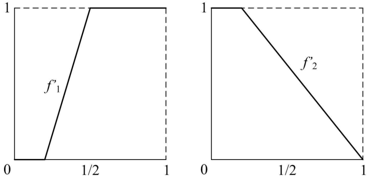

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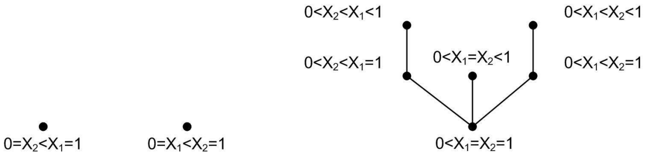

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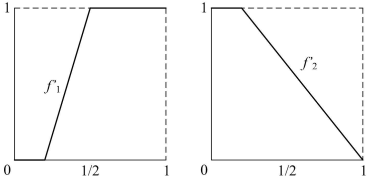

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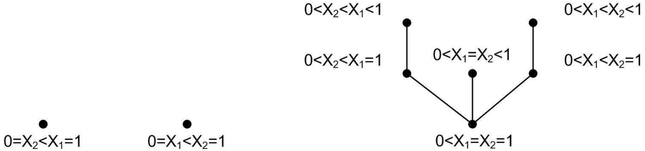


- Gödel logic cannot distinguish P' from P.

Weak Ruspini Partition

Let $\lambda : [0,1] \to [0,1]$ be an order preserving map such that $\lambda(0) = 0$ and $\lambda(1) = 1$, and let $t = \inf \lambda^{-1}(1)$. If the restriction of λ to $[0,t]$ is an order isomorphism between $[0,t]$ and $[0,1]$, we say λ is a *comparison map*.

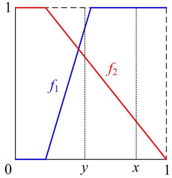
Definition

We say P is a *weak Ruspini partition* if for all $x \in [0,1]$, there exist $y \in [0,1]$, a comparison map λ, and an order isomorphism $\gamma : [0,1] \to [0,1]$ such that

(i) $\lambda(f_i(y)) = f_i(x)$, for all $i \in \underline{n}$.

(ii) $\sum_{i=1}^{n} \gamma(f_i(y)) = 1$.

Weak Ruspini Partition, an Example



A weak Ruspini partition $P = \{f_1, f_2\}$.

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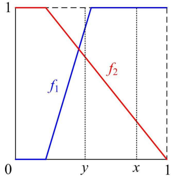

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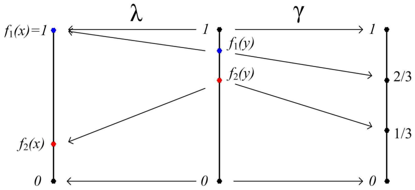
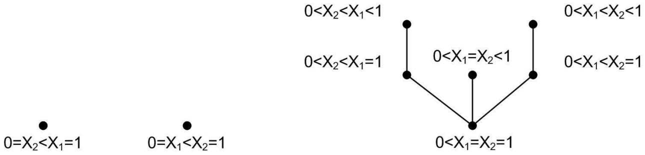


How the maps λ and γ work.

Ruspini Subforest

We denote by $\mathscr{R}_n$ the subforest of $\mathscr{F}_n$ obtained by removing from $\mathscr{F}_n$ the single tree having height 1, and the leaves of all the trees having height 2. We call $\mathscr{R}_n$ the *Ruspini forest*.



Definition

We say that a forest F is a *Ruspini subforest* if $F \subseteq \mathscr{R}_n$ and each leaf of F is a leaf of $\mathscr{R}_n$.

Ruspini Axiom

We define the *Ruspini axiom* $\rho_n = \alpha \vee \beta$, where

$$\alpha = \bigvee_{1 \le i < j \le n} (\neg\neg X_i \wedge \neg\neg X_j)\,,$$

$$\beta = \bigvee_{1 \le i \le n} (X_i \wedge \bigwedge_{1 \le j \ne i \le n} \neg X_j)\,.$$

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Lemma

$\mathscr{F}_{\rho_n} = \mathscr{R}_n\,.$

Main Result

Theorem

The following are equivalent.

(i) P is a weak Ruspini partition.

(ii) $\mathscr{F}(P)$ is a Ruspini subforest.

(iii) $\vdash \alpha \wedge \beta \wedge \gamma$, where

$$\alpha = (\alpha_P \to \rho_n),$$

$$\beta = \bigwedge_{r \in \mathrm{Root}(\mathscr{R}_n)} \bigwedge_{l \in \mathrm{Leaf}(r, \mathscr{R}_n)} ((\psi_l \to \alpha_P) \vee ((\psi_l \wedge \alpha_P) \to \psi_r)),$$

$$\gamma = \bigwedge_{r \in \mathrm{Root}(\mathscr{R}_n)} ((\psi_r \to \alpha_P) \to (\bigvee_{l \in \mathrm{Leaf}(r, \mathscr{R}_n)} (\psi_l \to \alpha_P))).$$

Moreover, for any Ruspini subforest F there exists a Ruspini partition $P' = \{f'_1, \dots, f'_n\}$, with $f'_i : [0,1] \to [0,1]$, such that $\mathscr{F}(P') = F$.

Conclusion

- Our analysis shows that Gödel logic does not have sufficient expressive power to capture the Ruspini condition.

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- Moreover, our Theorem shows that weak Ruspini partitions indeed are the best available approximations of Ruspini partitions in Gödel logic: for each weak Ruspini partition P, there exists a Ruspini partition P' such that there is no formula in Gödel logic telling P and P' apart.

Conclusion

- Our analysis shows that Gödel logic does not have sufficient expressive power to capture the Ruspini condition.
- However, we have proved that Gödel logic does capture the notion of weak Ruspini partition.
- Moreover, our Theorem shows that weak Ruspini partitions indeed are the best available approximations of Ruspini partitions in Gödel logic: for each weak Ruspini partition P, there exists a Ruspini partition P' such that there is no formula in Gödel logic telling P and P' apart.
- Up to Gödel equivalence, there is a finite number of weak Ruspini partitions of n elements. In our paper an exact formula to compute this number is given.

Further Work

- To analyze the expressibility of the Ruspini condition in other, more powerful, many-valued logics (*i.e.* Łukasiewicz logic).

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- To analyze the expressibility of the Ruspini condition in other, more powerful, many-valued logics (*i.e.* Łukasiewicz logic).
- To study, in a similar way, expressibility of other conditions on families of fuzzy sets (normality, convexity, . . .) .

Thank you

Thank you for your attention . . .