Combinatorial descriptions of products in the category of forests and open order-preserving maps

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(joint work with Ottavio M. D'Antona, and Vincenzo Marra)

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Basic notions.

A category of forests

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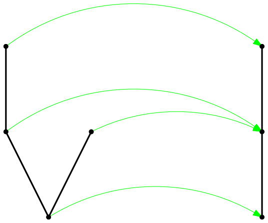

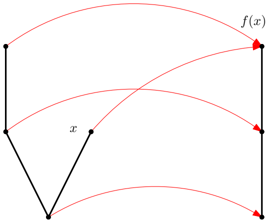

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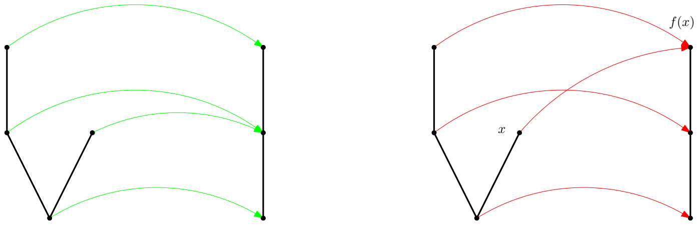

(The category of forests and open maps is dually equivalent to the category of finitely presented Gödel algebras.)

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Example. Let $\sigma = \{a|b\}$ and $\tau = \{x\}$. The merged shuffles of σ and τ are: $\{a|b|x\}$, $\{a|x|b\}$, $\{x|a|b\}$, $\{a|bx\}$, $\{ax|b\}$.

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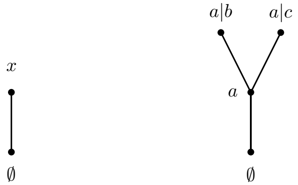

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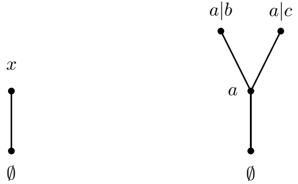

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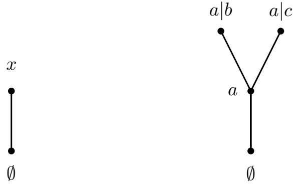

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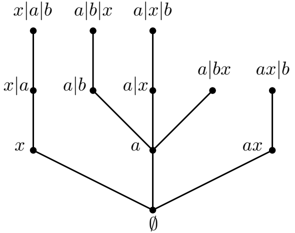

Product of forests.

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- Let $F = \{T_1, \ldots, T_r\}$ and $G = \{U_1, \ldots, U_s\}$ be forests. $F \times G = \{T_i \times U_j\}$, $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s\}$.

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How to compute the product of trees?

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Computing the product of trees (an example).

[D'Antona, O.M., and Marra, V., *Computing coproducts of finitely presented Gödel algebras*, Ann. Pure Appl. Logic 142 (2006), 202–211]

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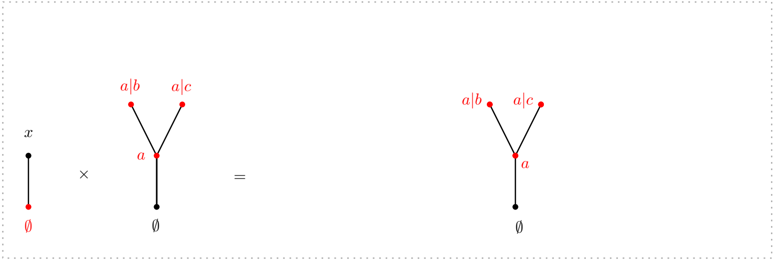

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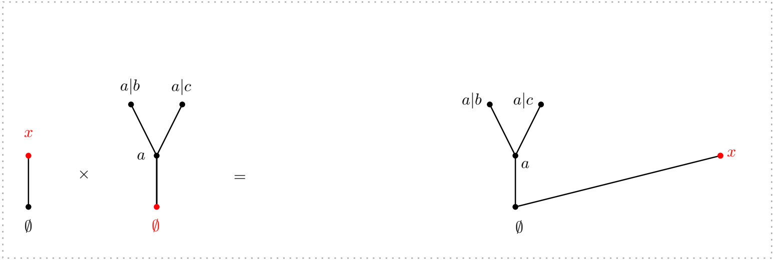

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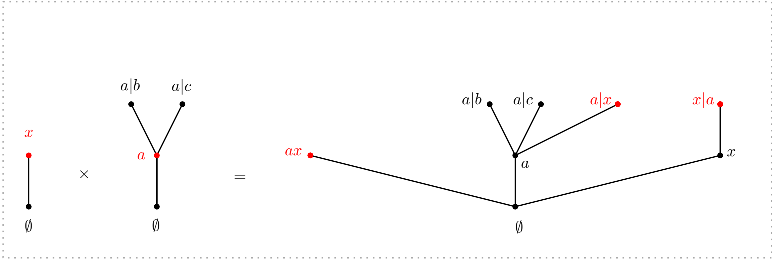

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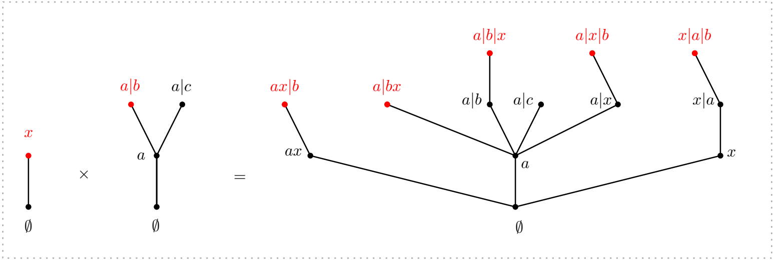

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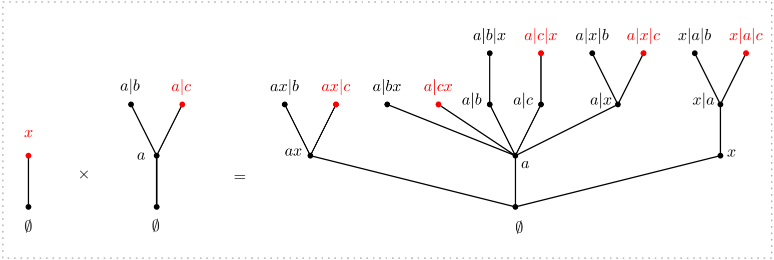

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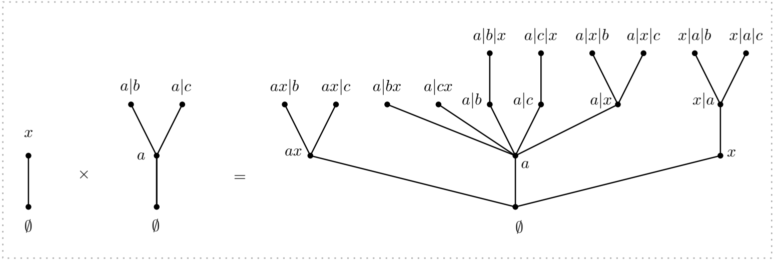

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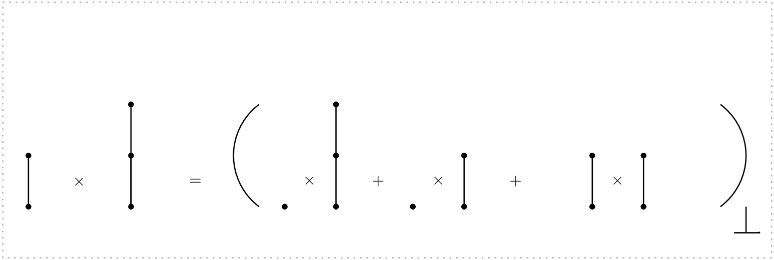

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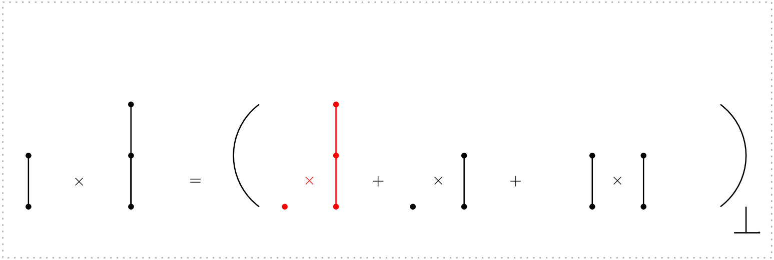

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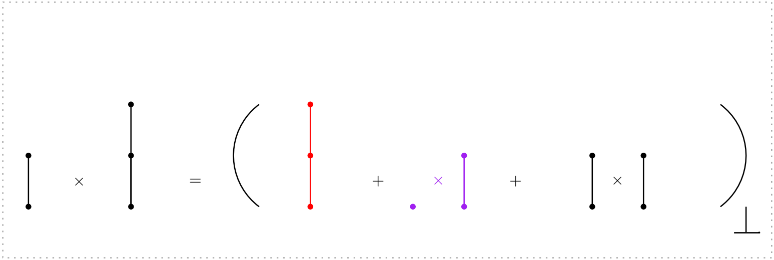

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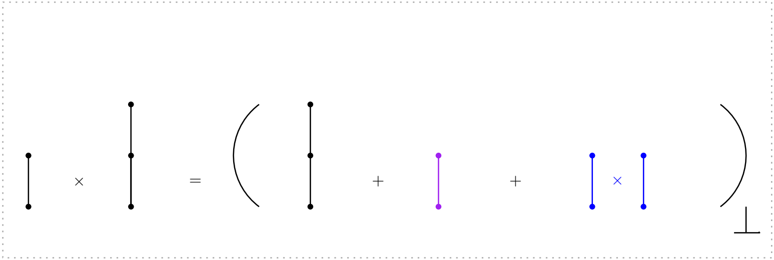

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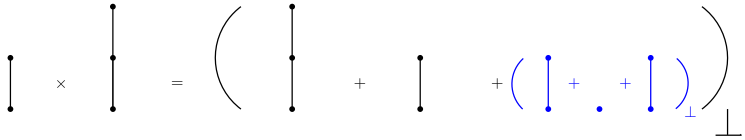

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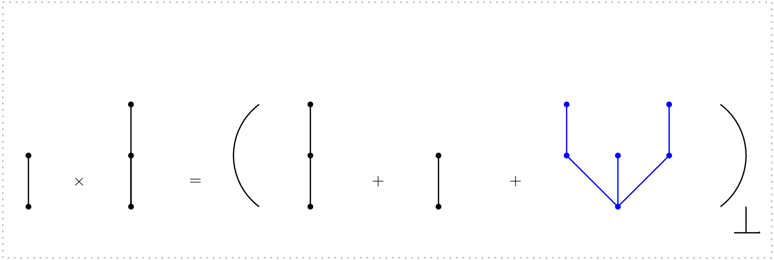

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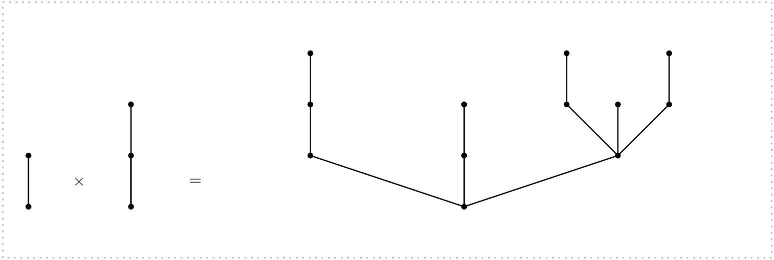

Enumeration.

Delannoy numbers

The Delannoy number $D_{n,m}$ counts the number of lattice paths from $(0,0)$ to (n,m) in which only East, North, and Northeast steps are allowed. Delannoy numbers satisfy the following recurrence relation.

$$D_{n,m} = D_{n-1,m} + D_{n,m-1} + D_{n-1,m-1}$$

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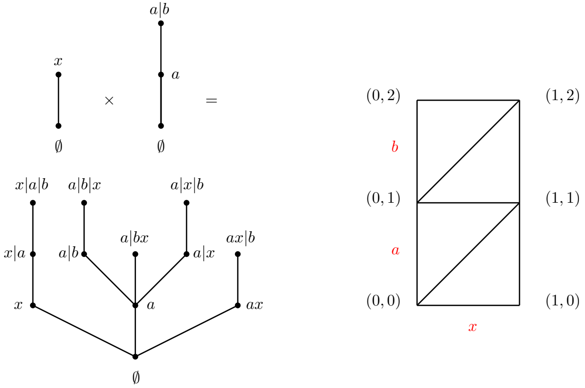
The following table shows some values of Delannoy numbers.

1	1	1	1	1	1	1	1
1	3	5	7	9	11	13	15
1	5	13	25	41	61	85	113
1	7	25	63	129	231	377	575
1	9	41	129	321	681	1289	2241
1	11	61	231	681	1683	3653	7183

Delannoy numbers and ordered partitions, a simple case

In a simple case, one can associate each element of the product with a Delannoy path, as follows.



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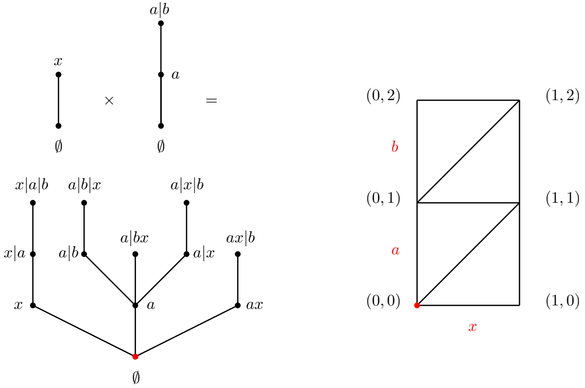

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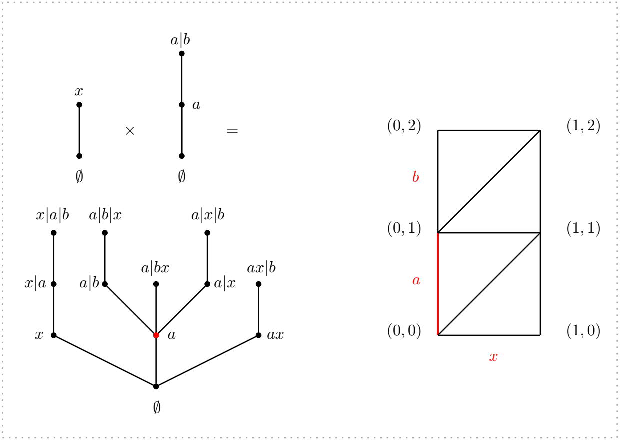

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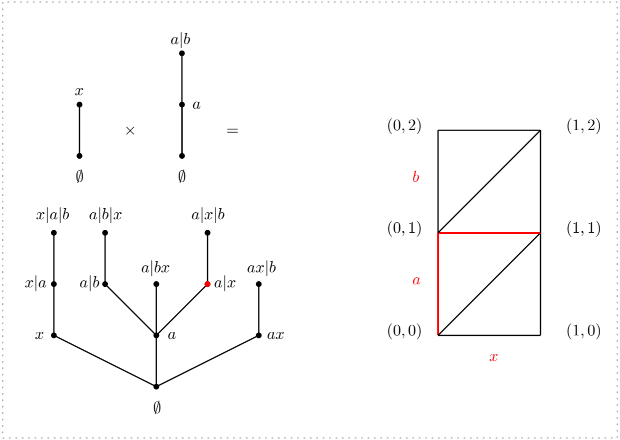

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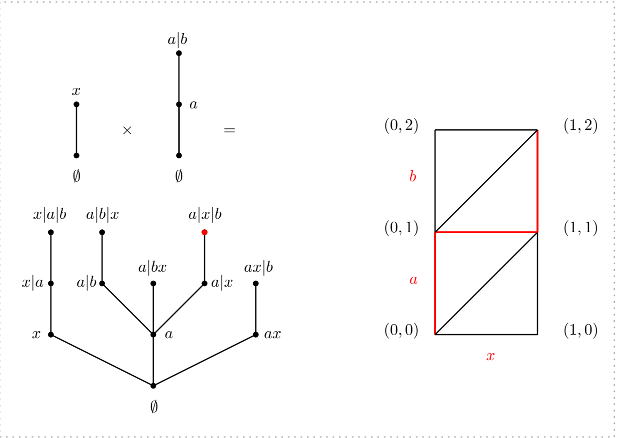

Delannoy numbers and ordered partitions, the general case

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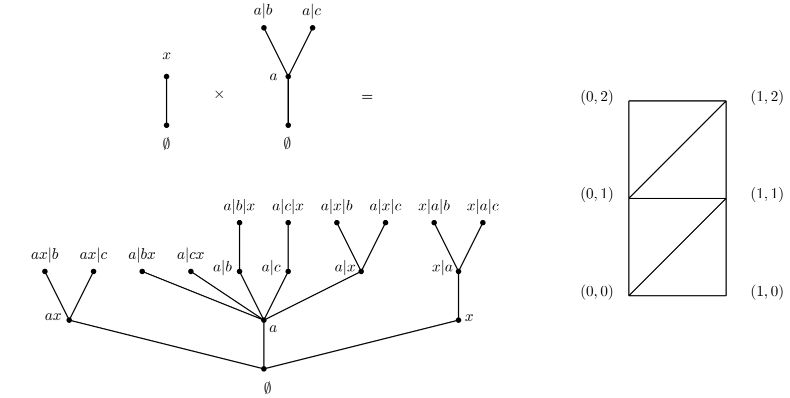

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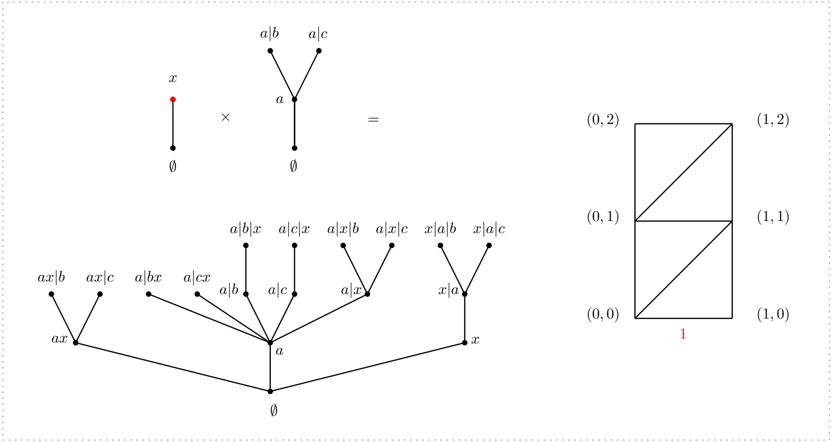

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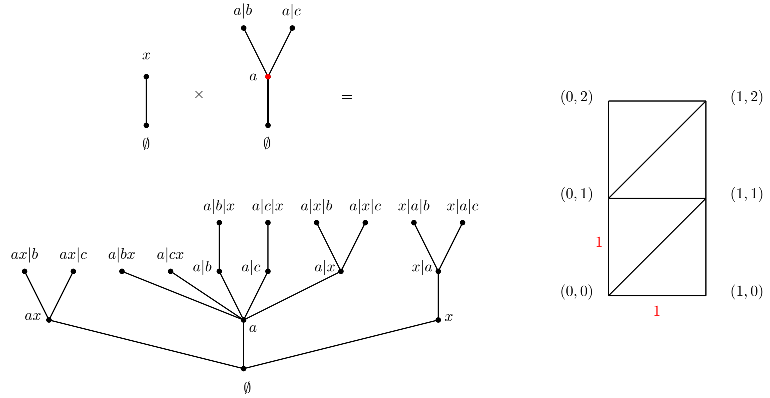

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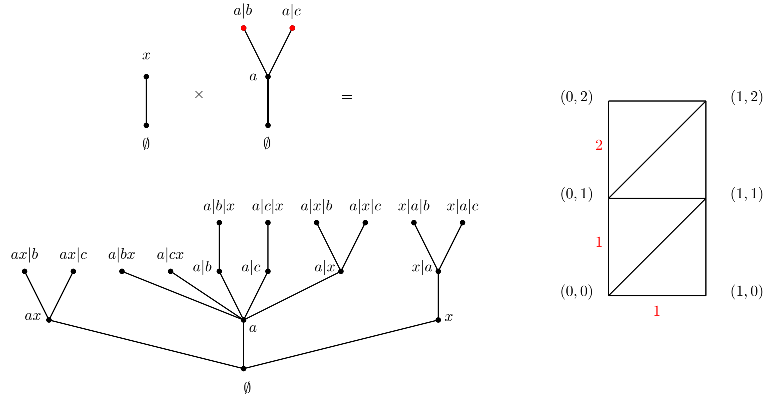

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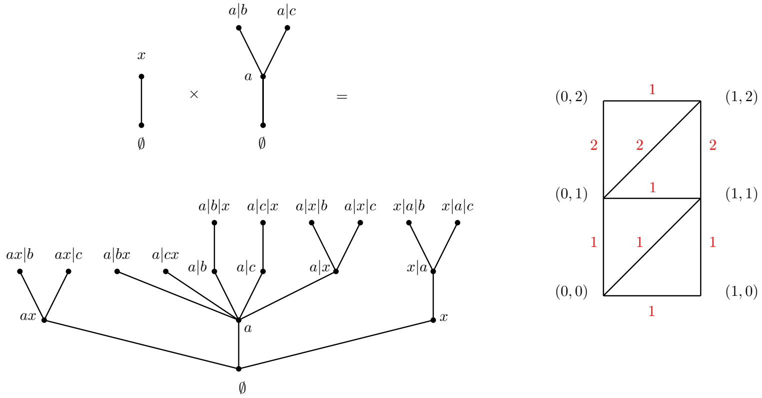

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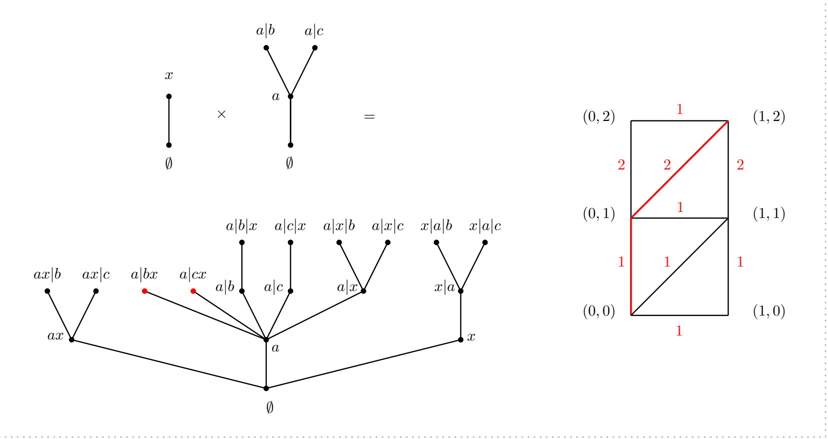

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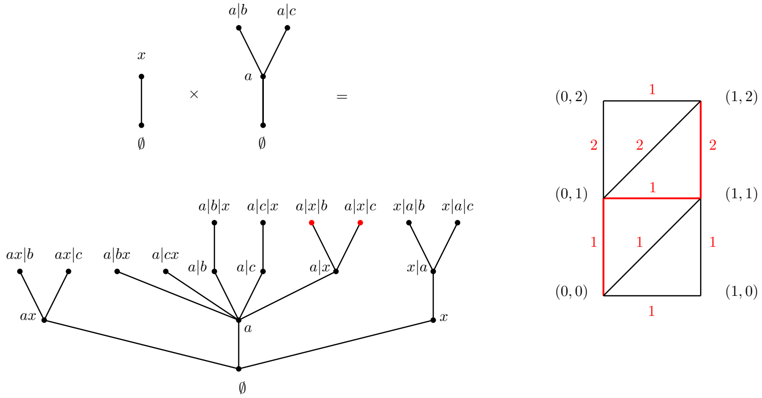

Formulæ, I

$$|T \times U| = \sum_{i \geq 0} \sum_{j \geq 0} t_i u_j D_{i,j} \,,$$

where t_i is the number of elements at level i of T, and u_j is the number of elements at level j of U.

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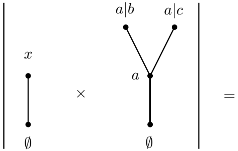
Example.



$$= 1 \cdot 1 \cdot D_{0,0} + 1 \cdot 1 \cdot D_{0,1} + 1 \cdot 2 \cdot D_{0,2} + 1 \cdot 1 \cdot D_{1,0} + 1 \cdot 1 \cdot D_{1,1} + 1 \cdot 2 \cdot D_{1,2} =$$
$$= 1 + 1 + 2 + 1 + 3 + 10 = 18.$$

Formulæ, II

Let L_T be the row vector of length $n = \mathtt{high}(T)$ containing the level numbers of the tree T. Let L_U be the column vector of height $m = \mathtt{high}(U)$ containing the level numbers of the tree U. Let $\mathcal{D}_{n,m}$ be the $n \times m$ matrix such that $\mathcal{D}[i,j] = D_{i,j}$, for each i, j. Then,

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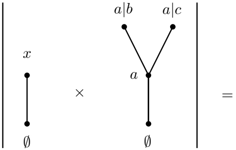

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 18$$

Formulæ, III

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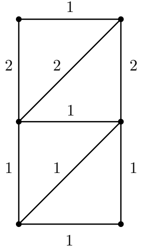
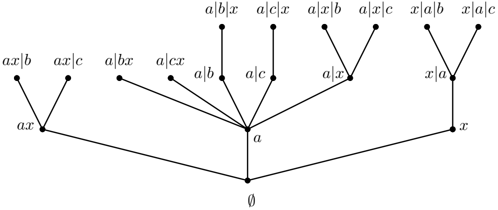
Example.





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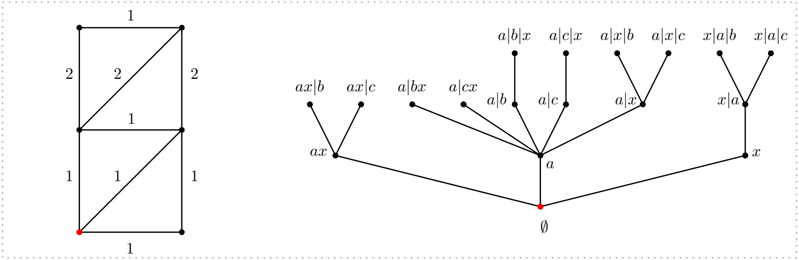

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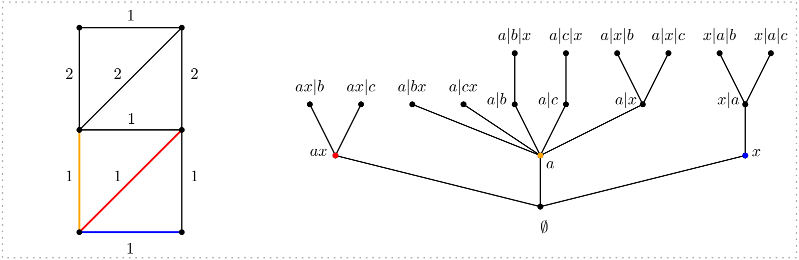

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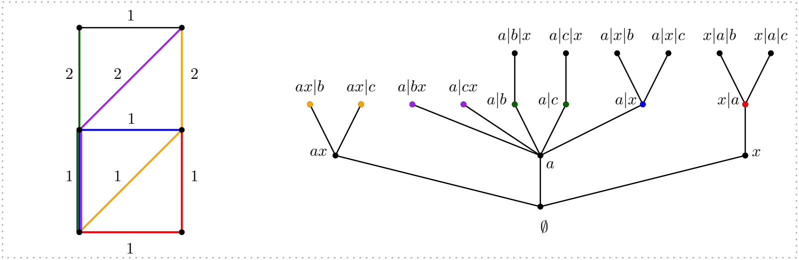

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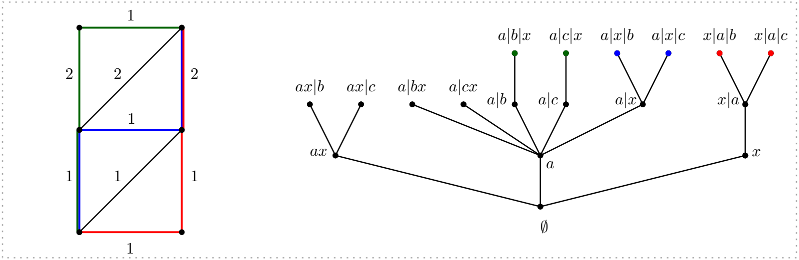
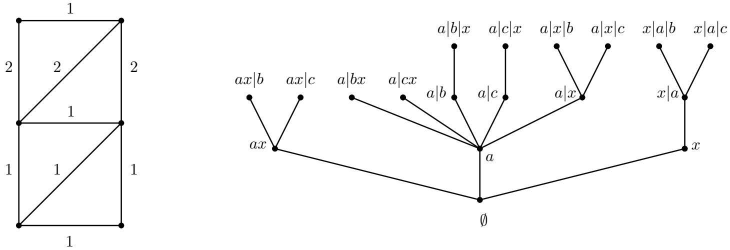


Formulæ, IV

Finally, we can count the number of leaves of the three $T \times U$ after a certain number p of *pruning*, as in the following example.

Example.



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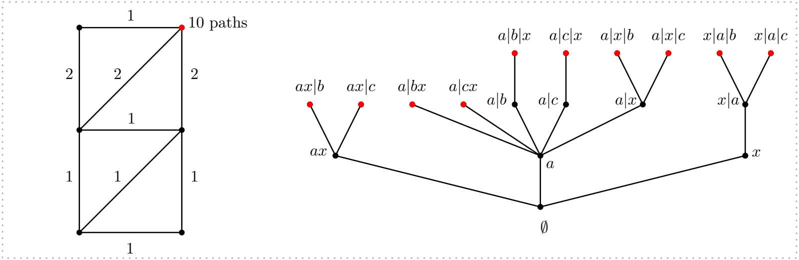

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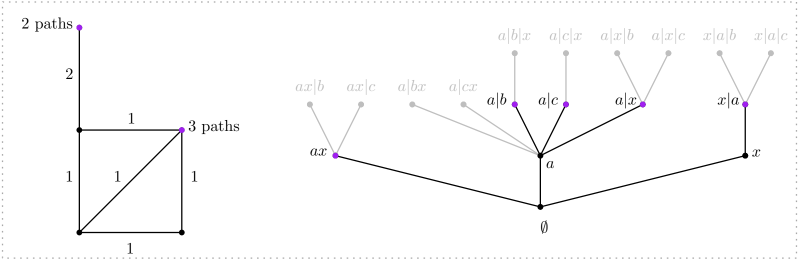

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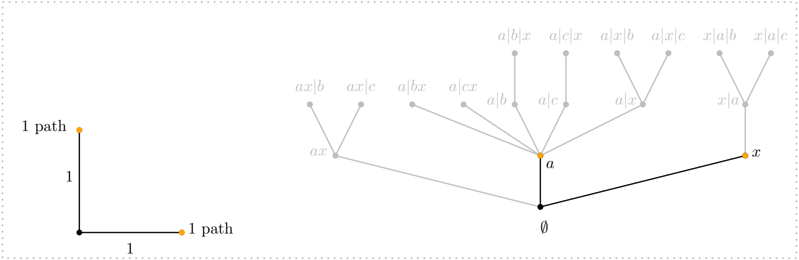

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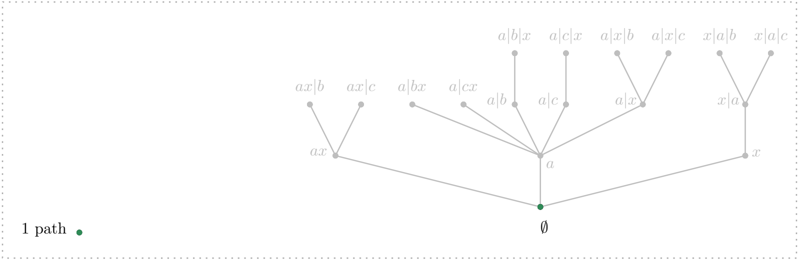

Thank you for your attention.