The independent subsets of powers of paths and cycles

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Outline

h-power of a paths and cycles — Definition

For a graph $\mathbf{G}$ we denote by $V(\mathbf{G})$ the set of its vertices, and by $E(\mathbf{G})$ the set of its edges.

h-power of a paths and cycles

For $n, h \geq 0$,

(i) the h-power of a path, denoted by $\mathbf{P}_n^{(h)}$ is a graph with n vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $(v_i, v_j) \in E(\mathbf{P}_n^{(h)})$ if and only if $|j - i| \leq h$;

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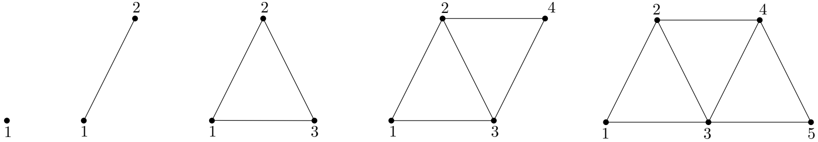
(ii) the h-power of a cycle, denoted by $\mathbf{Q}_n^{(h)}$ is a graph with n vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $(v_i, v_j) \in E(\mathbf{Q}_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$.

h-power of a paths and cycles — Example



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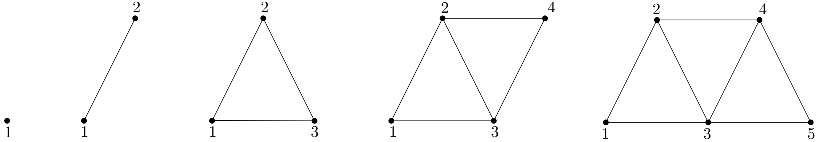

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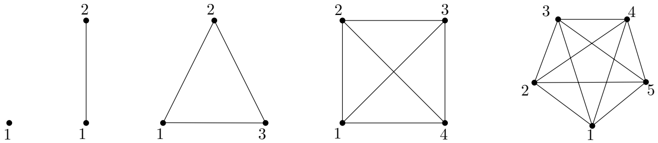

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In the first part of this presentation we evaluate $p_n^{(h)}$, i.e. the number of independent subsets of $\mathbf{P}_n^{(h)}$, and $H_n^{(h)}$, i.e. the number of edges of the Hasse diagram of the poset of independent subsets of $\mathbf{P}_n^{(h)}$ ordered by inclusion.

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In the second part we briefly consider the case of cycles.

Connections

Our main result is that, for $n, h \geq 0$, the sequence $H_n^{(h)}$ is obtained by convolving the sequence $\underbrace{1, \ldots, 1}_{h}, p_0^{(h)}, p_1^{(h)}, \ldots, p_1^{(n-h-1)}$ with itself.

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From a different perspective, this work could be seen as another generalization of the convolved Fibonacci sequence.

Evaluation of $p_{n,k}^{(h)}$

For $n, h, k \geq 0$, we denote by $p_{n,k}^{(h)}$ the number of independent k-subsets of $\mathbf{P}_n^{(h)}$. For convenience, we assume that $p_{n,k}^{(h)} = p_{0,k}^{(h)}$, whenever $n < 0$.

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Sketch of the proof. Establish a bijection between independent k-subset of $\mathbf{P}_n^{(h)}$ and k-subsets of a set with $(n - hk + h)$ elements.

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The coefficients $p_{n,k}^{(h)}$ also enjoy the property: $p_{n,k}^{(h)} = p_{n-k+1,k}^{(h-1)}$.

Evaluation of $p_n^{(h)}$

For $n, h \geq 0$, we denote by $p_n^{(h)}$ the number of all independent subsets of $\mathbf{P}_n^{(h)}$. We have:

$$p_n^{(h)} = \sum_{k \geq 0} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} .$$

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For $n, h \geq 0$,

$$p_n^{(h)} = \begin{cases} n+1 & \text{if } \; n \leq h+1\,, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } \; n > h+1\,. \end{cases}$$

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Sketch of the proof. Let $\mathcal{I}$ be the set of all independent subsets of $\mathbf{P}_n^{(h)}$, let $\mathcal{I}_{in}$ be the set of the independent subsets of $\mathbf{P}_n^{(h)}$ that contain v_n, and let $\mathcal{I}_{out} = \mathcal{I} \setminus \mathcal{I}_{in}$. ...

The poset of independent subsets of power of paths

Let $\mathbf{H}_n^{(h)}$ be the Hasse diagram of the poset of independent subsets of $\mathbf{P}_n^{(h)}$ ordered by inclusion, and $H_n^{(h)}$ the number of the edges of the diagram.

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Noting that in $\mathbf{H}_n^{(h)}$ each non-empty independent k-subset covers exactly k independent $(k-1)$-subsets, we can write

$$H_n^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} k p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} k \binom{n - hk + h}{k} .$$

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For $h \geq 0$, we define the h-Fibonacci sequence $\mathcal{F}^{(h)} = \{F_n^{(h)}\}_{n \geq 1}$ whose elements are

$$F_n^{(h)} = \begin{cases} 1 & \text{if } 1 \leq n \leq h+1\,, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h+1. \end{cases}$$

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- $\mathcal{F}^{(0)} = 1, 2, 4, \ldots, 2^n, \ldots;$
- $\mathcal{F}^{(1)}$ is the Fibonacci sequence;
- more generally, $\mathcal{F}^{(h)} = \underbrace{1, \ldots, 1}_{h}, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots.$

Main result

We use of the discrete convolution operation $*$, as follows.

$$\left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n) \doteq \sum_{i=1}^{n} F_i^{(h)} \cdot F_{n-i+1}^{(h)}$$

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Main result

For $n, h \geq 0$, the following holds.

$$H_n^{(h)} = \left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n).$$

Proof of main result — 1

Let $T_{k,i}^{(n,h)}$ be the number of independent k-subsets of $\mathbf{P}_n^{(h)}$ containing the vertex v_i.

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For $n, h, k \geq 0$, and $1 \leq i \leq n$,

$$T_{k,i}^{(n,h)} = \sum_{0 \leq r+s=k-1} p_{i-h-1,r}^{(h)} \cdot p_{n-i-h,s}^{(h)}\,.$$

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Lemma

For positive n,

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} .$$

Proof of main result — 2

Sketch of the proof. Using

$$\sum_{k=1}^{\lceil n/(h+1)\rceil} T_{k,i}^{(n,h)} = p_{i-h-1}^{(h)} \cdot p_{n-h-i}^{(h)},$$

and the Lemmas,

Proof of main result — 2

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and the Lemmas, we obtain

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1)\rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \sum_{k=1}^{\lceil n/(h+1)\rceil} T_{k,i}^{(n,h)} =$$

$$= \sum_{i=1}^{n} p_{i-h-1}^{(h)} \cdot p_{n-h-i}^{(h)} = \sum_{i=1}^{n} F_i^{(h)} \cdot F_{n-i+1}^{(h)}.$$

Some values of $H_n^{(h)}$

The table below supplies a few values of $H_n^{(h)}$.

$H_n^{(h)}$	$n=0$	1	2	3	4	5	6	7	8	9	10	11
$H_n^{(0)}$	0	1	4	12	32	80	192	448	1024	2304	5120	11264
$H_n^{(1)}$	0	1	2	5	10	20	38	71	130	235	420	744
$H_n^{(2)}$	0	1	2	3	6	11	18	30	50	81	130	208
$H_n^{(3)}$	0	1	2	3	4	7	12	19	28	42	64	97
$H_n^{(4)}$	0	1	2	3	4	5	8	13	20	29	40	56
$H_n^{(5)}$	0	1	2	3	4	5	6	9	14	21	30	41

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We denote by $q_{n,k}^{(h)}$ the number of independent k-subsets of $\mathbf{Q}_n^{(h)}$. For $n, h, k \geq 0$,

$$q_{n,k}^{(h)} = \frac{n}{k}\binom{n-hk-1}{k-1} . \qquad \left[p_{n,k}^{(h)} = \binom{n-hk+h}{k}\right]$$

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For $n, h \geq 0$, we denote by $q_n^{(h)}$ the number of all independent subsets of $\mathbf{Q}_n^{(h)}$: $q_n^{(h)} = \sum_{k=0}^{\lfloor n/(h+1)\rfloor} q_{n,k}^{(h)}$.

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$$q_n^{(h)} = \begin{cases} n+1 & \text{if } \; n \leq 2h+1 \, , \\ q_{n-1}^{(h)} + q_{n-h-1}^{(h)} & \text{if } \; n > 2h+1 . \end{cases}$$

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We immediately provide a formula for $L_n^{(h)}$:

$$L_n^{(h)} = \sum_{k=0}^{\lfloor n/(h+1) \rfloor} k q_{n,k}^{(h)} = n \sum_{k=0}^{\lfloor n/(h+1) \rfloor} \binom{n-hk-1}{k-1}$$

$$\left[H_n^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} k p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} k \binom{n-hk+h}{k} \right]$$

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The main result has no analog in the case of cycles.

Thank you for your attention.