The Euler Characteristic of a formula in many-valued logic

Pietro Codara  Ottavio M. D’Antona  Vincenzo Marra

Dipartimento di Informatica e Comunicazione, Università degli Studi di Milano

Presenting author: Pietro Codara

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Outline

1. The lattice-theoretic Euler Characteristic
2. Euler Characteristic of a formula in classical propositional logic
3. Euler Characteristic of a formula in Gödel logic
4. Further research
Let $L$ be a (bounded) distributive lattice whose bottom element is denoted $\bot$. A function $\nu: L \to \mathbb{R}$ is a **valuation** if it satisfies $\nu(\bot) = 0$, and

$$\nu(x) + \nu(y) = \nu(x \lor y) + \nu(x \land y)$$

for all $x, y \in L$. 

**Lemma**

Every valuation on a finite distributive lattice $L$ is uniquely determined by its values at the join-irreducibles of $L$. Recall that $x \in L$ is join-irreducible if it is not the bottom of $L$, and $x = y \lor z$ implies $x = y$ or $x = z$ for all $y, z \in L$. 

**Valuation**

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The Euler-Klee-Rota lattice-theoretic characteristic, official definition:

(V. Klee 1963; G.-C. Rota 1974)

**Euler characteristic**

The Euler characteristic of a finite distributive lattice $L$ is the unique valuation $\chi: L \rightarrow \mathbb{R}$ such that $\chi(x) = 1$ for any join-irreducible element $x \in L$. 
Euler characteristic of a classical formula

For an integer \( n \geq 0 \), let \( \text{FORM}_n \) denote the set of formulæ in classical (propositional) logic over the atomic propositions \( X_1, \ldots, X_n \) and the logical constant \( \bot \) (\textit{falsum}).
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- Then we say that the Euler characteristic of $\varphi$ is $\chi([\varphi]_\equiv)$. 
Euler characteristic of Boolean algebras

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- Since atoms of \( \text{FORM}_n/\equiv \) are in natural bijections with assignments of truth values \( \mu: \{X_1, \ldots, X_n\} \to \{0, 1\} \), we have:
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\[ \chi([\varphi]_{\equiv}) \text{ is the number of assignments that satisfy } \varphi. \]
Gödel logic

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An assignment is a function $\mu: \text{FORM} \to [0, 1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha, \beta \in \text{FORM},$

$$\mu(\alpha \land \beta) = \min\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \lor \beta) = \max\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \rightarrow \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases}$$

and $\mu(\neg \alpha) = \mu(\alpha \rightarrow \bot)$, $\mu(\bot) = 0$, $\mu(\top) = 1$. 
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\end{align*}

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A tautology is a formula $\alpha$ such that $\mu(\alpha) = 1$ for every assignment $\mu$. 
Gödel algebras

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity axiom

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They provide the equivalent algebraic semantics of Gödel logic. For an integer $n \geq 0$, let us write $G_n$ for the Tarski-Lindenbaum algebra of Gödel logic over the variables $X_1, \ldots, X_n$, that is, the algebra $\text{ FORM}_n / \equiv$, where $\equiv$ is the logical equivalence between formulæ.
We shall use Gödel \((k + 1)\)-valued logic, written \(G_{k+1}\), for an integer \(k \geq 1\).
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\(G_{k+1}\) is obtained from \(G_{\infty}\), Gödel (infinite-valued) logic recalled above, by restricting assignments to those taking values in the set

\[V_{k+1} = \{0 = \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \frac{k}{k} = 1\} \subseteq [0, 1],\]

that is, to \((k + 1)\)-valued assignments.
Euler characteristic of a formula Gödel logic

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Does this notion have any logical content?
Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula \( \varphi \in \text{FORM}_n \), written \( \chi(\varphi) \), is the number \( \chi([\varphi]_{\equiv}) \), where \( \chi \) is the Euler characteristic of the finite distributive lattice \( G_n \).

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**Theorem**

Fix an integer \( n \geq 1 \). For any formula \( \varphi \in \text{FORM}_n \), the Euler characteristic \( \chi(\varphi) \) equals the number of Boolean assignments \( \mu : \text{FORM}_n \to [0,1] \) such that \( \mu(\varphi) = 1 \).
### Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \text{FORM}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where $\chi$ is the Euler characteristic of the finite distributive lattice $G_n$.

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### Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \text{FORM}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu: \text{FORM}_n \to [0, 1]$ such that $\mu(\varphi) = 1$.

In the sense given by this result, the characteristic of a formula as defined above is a classical notion – it will not distinguish, for instance, classical from non-classical tautologies.
Generalised Euler characteristic of a formula in Gödel logic

For a join-irreducible $g \in \mathcal{G}_n$, say $g$ has height $h(g)$ if the (unique) chain of join-irreducibles below $g$ in $\mathcal{G}_n$ has cardinality $h(g)$. 
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**Generalised Euler characteristic**

Fix integers $n, k \geq 1$. We write $\chi_k : \mathcal{G}_n \rightarrow \mathbb{R}$ for the unique valuation on $\mathcal{G}_n$ that satisfies

$$\chi_k(g) = \min \{ h(g), k \}$$

for each join-irreducible element $g \in \mathcal{G}_n$. Further, if $\varphi \in \text{FORM}_n$, we define $\chi_k(\varphi) = \chi_k([\varphi]_\equiv)$.

It turns out that $\chi_k$ is a “$k$-valued characteristic”, as we proceed to show.
Our next aim is to relate $\chi_k$ with (not necessarily Boolean) $[0, 1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu: \{X_1\} \rightarrow [0, 1]$. However, in Gödel logic this fact is quite misleading, and there is the following important reduction to finiteness.
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**n-equivalence**

Fix integers $n, k \geq 1$. We say that two $(k + 1)$-valued assignments $\mu$ and $\nu$ are *equivalent over the first $n$ variables*, or just *n-equivalent*, if and only if for all formulæ $\varphi(X_1, \ldots, X_n)$ of $G_{k+1}$, $\mu(\varphi) = 1$ if and only if $\nu(\varphi) = 1$.

The same definition can be given, *mutatis mutandis*, for $G_\infty$. 
Reduction to finitely many possible worlds

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where $P(n, k) = \sum_{i=1}^{k} \sum_{j=0}^{n} \binom{n}{j} T(j, i)$,
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This many: $P(n, n + 1)$, where

$$P(n, k) = \sum_{i=1}^{k} \sum_{j=0}^{n} \binom{n}{j} T(j, i),$$

and

$$T(n, k) = \begin{cases} 
1 & \text{if } k = 1, \\
0 & \text{if } k > n + 1, \\
\sum_{i=1}^{n} \binom{n}{i} T(n - i, k - 1) & \text{otherwise}.
\end{cases}$$
The number $P(n, k)$ of distinct equivalence classes of $(k + 1)$-valued assignments over $n$ variables.
Main result

Theorem

Fix integers \( n, k \geq 1 \), and a formula \( \varphi \in \text{FORM}_n \).

\[ \chi_k(\varphi) \] equals the number of \((k + 1)\)-valued assignments \( \mu: \text{FORM}_n \rightarrow [0, 1] \) such that \( \mu(\varphi) = 1 \), up to \( n \)-equivalence.
Main result

**Theorem**

Fix integers $n, k \geq 1$, and a formula $\varphi \in \text{FORM}_n$.

1. $\chi_k(\varphi)$ equals the number of $(k + 1)$-valued assignments $\mu: \text{FORM}_n \to [0, 1]$ such that $\mu(\varphi) = 1$, up to $n$-equivalence.

2. $\varphi$ is a tautology in $\mathbb{G}_{k+1}$ if and only if $\chi_k(\varphi) = P(n, k)$. 
Main result

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1. $\chi_k(\varphi)$ equals the number of $(k + 1)$-valued assignments $\mu: \text{FORM}_n \to [0, 1]$ such that $\mu(\varphi) = 1$, up to $n$-equivalence.

2. $\varphi$ is a tautology in $G_{k+1}$ if and only if $\chi_k(\varphi) = P(n, k)$.

3. $\varphi$ is a tautology in $G_{\infty}$ if and only if it is a tautology in $G_{n+2}$ if and only if $\chi_{n+1}(\varphi) = P(n, n + 1)$.
Example

The Gödel algebra $G_1$, and the values of $\chi = \chi_1 : G_1 \to \mathbb{R}$ and $\chi_2 : G_1 \to \mathbb{R}$. 
Further research

- Study the logical content of the Euler Characteristic for Łukasiewicz logic, and other many-valued logics.

- Find an appropriate version of the Euler characteristic for Łukasiewicz logic, and other many-valued logics.

- More generally: Develop the general theory of valuations over Gödel and MV-algebras.
Thank you for your attention.