The Euler Characteristic of a formula in many-valued logic

Pietro Codara Ottavio M. D'Antona Vincenzo Marra

Dipartimento di Informatica e Comunicazione, Università degli Studi di Milano

Presenting author: Pietro Codara

Combinatorics 2010, Verbania, 2 July 2010

Outline

1. The lattice-theoretic Euler Characteristic
2. Euler Characteristic of a formula in classical propositional logic
3. Euler Characteristic of a formula in Gödel logic
4. Further research

The Euler-Klee-Rota lattice-theoretic characteristic

Valuation

Let L be a (bounded) distributive lattice whose bottom element is denoted $\perp$. A function $\nu \colon L \to \mathbb{R}$ is a valuation if it satisfies $\nu(\perp) = 0$, and

$$\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y)$$

for all $x, y \in L$.

The Euler-Klee-Rota lattice-theoretic characteristic

Valuation

Let L be a (bounded) distributive lattice whose bottom element is denoted $\perp$. A function $\nu\colon L \to \mathbb{R}$ is a valuation if it satisfies $\nu(\perp) = 0$, and

$$\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y)$$

for all $x, y \in L$.

Lemma

Every valuation on a finite distributive lattice L is uniquely determined by its values at the join-irreducibles of L.

Recall that $x \in L$ is *join-irreducible* if it is not the bottom of L, and $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$.

The Euler-Klee-Rota lattice-theoretic characteristic, official definition:

(V. Klee 1963; G.-C. Rota 1974)

Euler characteristic

The Euler characteristic of a finite distributive lattice L is the unique valuation $\chi\colon L \to \mathbb{R}$ such that $\chi(x) = 1$ for any join-irreducible element $x \in L$.

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \textsc{Form}$?

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let FORM_n denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \mathrm{FORM}$?
- Writing $\equiv$ for the relation of logical equivalence, the quotient set $\mathrm{FORM}_n / \equiv$ is naturally a Boolean algebra.

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \textsc{Form}$?
- Writing $\equiv$ for the relation of logical equivalence, the quotient set $\textsc{Form}_n / \equiv$ is naturally a Boolean algebra.
- So we can consider valuations on $\textsc{Form}_n / \equiv$. In particular, let χ be the Euler(-Klee-Rota) characteristic of $\textsc{Form}_n / \equiv$.

Euler characteristic of a classical formula

- For an integer $n \geq 0$, let $\textsc{Form}_n$ denote the set of formulæ in classical (propositional) logic over the atomic propositions $X_1, \ldots, X_n$ and the logical constant $\perp$ (*falsum*).
- **Question**: Is there a sensible notion of *Euler characteristic* for a formula $\varphi \in \textsc{Form}$?
- Writing $\equiv$ for the relation of logical equivalence, the quotient set $\textsc{Form}_n/\equiv$ is naturally a Boolean algebra.
- So we can consider valuations on $\textsc{Form}_n/\equiv$. In particular, let χ be the Euler(-Klee-Rota) characteristic of $\textsc{Form}_n/\equiv$.
- Then we say that *the Euler characteristic of* φ *is* $\chi([\varphi]_{\equiv})$.

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)
- In other words, if B is canonically represented as the Boolean algebra of all subsets of a set (=the set of atoms of B), then $\chi(S)$=cardinality of S for all $S \in B$.

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)
- In other words, if B is canonically represented as the Boolean algebra of all subsets of a set (=the set of atoms of B), then $\chi(S)$=cardinality of S for all $S \in B$.
- Since atoms of $\textsc{Form}_n/\equiv$ are in natural bijections with assignments of truth values $\mu\colon \{X_1, \ldots, X_n\} \to \{0, 1\}$, we have:

Euler characteristic of Boolean algebras

- In finite Boolean algebras, join-irreducible=atom.
- So if B is a finite Boolean algebra, and $x \in B$ is the join of n atoms, we have $\chi(x) = n$ by the valuation property. (The characteristic is additive over disjoint elements.)
- In other words, if B is canonically represented as the Boolean algebra of all subsets of a set (=the set of atoms of B), then $\chi(S)$=cardinality of S for all $S \in B$.
- Since atoms of $\textsc{Form}_n/\equiv$ are in natural bijections with assignments of truth values $\mu\colon \{X_1, \dots, X_n\} \to \{0, 1\}$, we have:

$\chi([\varphi]_\equiv)$ *is the number of assignments that satisfy* φ.

Gödel logic

Gödel logic $\mathbb{G}_\infty$ can be semantically defined as a many-valued logic.

Gödel logic

Gödel logic $\mathbb{G}_\infty$ can be semantically defined as a many-valued logic.

Let FORM be the set of formulæ over propositional variables $X_1, X_2, \ldots$ in the language $\wedge, \vee, \to, \neg, \bot, \top$.

Gödel logic

Gödel logic $\mathbb{G}_\infty$ can be semantically defined as a many-valued logic.

Let FORM be the set of formulæ over propositional variables $X_1, X_2, \ldots$ in the language $\wedge, \vee, \rightarrow, \neg, \bot, \top$.

An assignment is a function $\mu\colon \text{FORM} \rightarrow [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha,\ \beta \in \text{FORM}$,

$$\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \vee \beta) = \max\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \rightarrow \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases}$$

and $\mu(\neg\alpha) = \mu(\alpha \rightarrow \bot)$, $\mu(\bot) = 0$, $\mu(\top) = 1$.

Gödel logic

Gödel logic $\mathbb{G}_\infty$ can be semantically defined as a many-valued logic.

Let $\textsc{Form}$ be the set of formulæ over propositional variables $X_1, X_2, \ldots$ in the language $\wedge, \vee, \rightarrow, \neg, \perp, \top$.

An assignment is a function $\mu \colon \textsc{Form} \rightarrow [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha,\ \beta \in \textsc{Form}$,

$$\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \vee \beta) = \max\{\mu(\alpha), \mu(\beta)\}$$

$$\mu(\alpha \rightarrow \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases}$$

and $\mu(\neg\alpha) = \mu(\alpha \rightarrow \perp)$, $\mu(\perp) = 0$, $\mu(\top) = 1$.

A tautology is a formula α such that $\mu(\alpha) = 1$ for every assignment μ.

Gödel algebras

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity axiom

$$(x \to y) \vee (y \to x) = \top \ .$$

Gödel algebras

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity axiom

$$(x \to y) \vee (y \to x) = \top \ .$$

They provide the equivalent algebraic semantics of Gödel logic.

Gödel algebras

Gödel algebras are Heyting algebras (=Tarski-Lindenbaum algebras of intuitionistic propositional calculus) satisfying the prelinearity axiom

$$(x \to y) \vee (y \to x) = \top \; .$$

They provide the equivalent algebraic semantics of Gödel logic. For an integer $n \geq 0$, let us write $\mathscr{G}_n$ for the Tarski-Lindenbaum algebra of Gödel logic over the variables $X_1, \ldots, X_n$, that is, the algebra $\textsc{Form}_n / \equiv$, where $\equiv$ is the logical equivalence between formulæ.

Gödel $(k+1)$-valued logic

We shall use Gödel $(k+1)$-valued logic, written $\mathbb{G}_{k+1}$, for an integer $k \geq 1$.

Gödel $(k+1)$-valued logic

We shall use Gödel $(k+1)$-valued logic, written $\mathbb{G}_{k+1}$, for an integer $k \geq 1$.

$\mathbb{G}_{k+1}$ is obtained from $\mathbb{G}_\infty$, Gödel (infinite-valued) logic recalled above, by restricting assignments to those taking values in the set

$$V_{k+1} = \{0 = \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \frac{k}{k} = 1\} \subseteq [0,1] \ ,$$

that is, to $(k+1)$-*valued assignments.*

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_{\equiv})$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Does this notion have any logical content?

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_{\equiv})$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Does this notion have any logical content?

Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \textsc{Form}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$.

Euler characteristic of a formula Gödel logic

Euler characteristic of a formula in Gödel logic

The Euler characteristic of a formula $\varphi \in \textsc{Form}_n$, written $\chi(\varphi)$, is the number $\chi([\varphi]_\equiv)$, where χ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

Does this notion have any logical content?

Theorem

Fix an integer $n \geq 1$. For any formula $\varphi \in \textsc{Form}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$.

In the sense given by this result, the characteristic of a formula as defined above is a classical notion – it will not distinguish, for instance, classical from non-classical tautologies.

Generalised Euler characteristic of a formula in Gödel logic

For a join-irreducible $g \in \mathscr{G}_n$, say g has height $h(g)$ if the (unique) chain of join-irreducibles below g in $\mathscr{G}_n$ has cardinality $h(g)$.

Generalised Euler characteristic of a formula in Gödel logic

For a join-irreducible $g \in \mathscr{G}_n$, say g has height $h(g)$ if the (unique) chain of join-irreducibles below g in $\mathscr{G}_n$ has cardinality $h(g)$.

Generalised Euler characteristic

Fix integers $n, k \geq 1$. We write $\chi_k \colon \mathscr{G}_n \longrightarrow \mathbb{R}$ for the unique valuation on $\mathscr{G}_n$ that satisfies

$$\chi_k(g) = \min\{h(g), k\}$$

for each join-irreducible element $g \in \mathscr{G}_n$. Further, if $\varphi \in \textsc{Form}_n$, we define $\chi_k(\varphi) = \chi_k([\varphi]_\equiv)$.

It turns out that χ_k is a "k-valued characteristic", as we proceed to show.

n-equivalence

Our next aim is to relate χ_k with (not necessarily Boolean) $[0,1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu \colon \{X_1\} \to [0,1]$. However, in Gödel logic this fact is quite misleading, and there is the following important reduction to finiteness.

n-equivalence

Our next aim is to relate χ_k with (not necessarily Boolean) $[0, 1]$-valued assignments. In general, even if $n = 1$ and the language boils down to $\{X_1\}$, there are uncountably many assignments $\mu\colon \{X_1\} \to [0, 1]$. However, in Gödel logic this fact is quite misleading, and there is the following important reduction to finiteness.

n-equivalence

Fix integers $n, k \geq 1$. We say that two $(k + 1)$-valued assignments μ and ν are *equivalent over the first n variables*, or just *n-equivalent*, if and only if for all formulæ $\varphi(X_1, \ldots, X_n)$ of $\mathbb{G}_{k+1}$, $\mu(\varphi) = 1$ if and only if $\nu(\varphi) = 1$.
The same definition can be given, *mutatis mutandis*, for $\mathbb{G}_\infty$.

Reduction to finitely many possible worlds

In $\mathscr{G}_\infty$, there are only finitely many equivalence classes of $[0, 1]$-valued assignments to n variables.

Reduction to finitely many possible worlds

In $\mathscr{G}_\infty$, there are only finitely many equivalence classes of $[0,1]$-valued assignments to n variables.

How many, exactly?

Reduction to finitely many possible worlds

In $\mathscr{G}_\infty$, there are only finitely many equivalence classes of $[0,1]$-valued assignments to n variables.

How many, exactly?

This many: $P(n, n+1)$,

Reduction to finitely many possible worlds

In $\mathscr{G}_\infty$, there are only finitely many equivalence classes of $[0,1]$-valued assignments to n variables.

How many, exactly?

This many: $P(n, n+1)$,

where $P(n,k) = \sum_{i=1}^{k} \sum_{j=0}^{n} \binom{n}{j} T(j,i)$,

Reduction to finitely many possible worlds

In $\mathscr{G}_\infty$, there are only finitely many equivalence classes of $[0,1]$-valued assignments to n variables.

How many, exactly?

This many: $P(n, n+1)$,

where $P(n,k) = \sum_{i=1}^{k} \sum_{j=0}^{n} \binom{n}{j} T(j,i)$, and

$$T(n,k) = \begin{cases} 1 & \text{if } k = 1\,, \\ 0 & \text{if } k > n+1\,, \\ \displaystyle\sum_{i=1}^{n} \binom{n}{i} T(n-i, k-1) & \text{otherwise}\,. \end{cases}$$

Table

	k=1	2	3	4	5	6	7
n=1	2	3	3	3	3	3	3
2	4	9	11	11	11	11	11
3	8	27	45	51	51	51	51
4	16	81	191	275	299	299	299
5	32	243	813	1563	2043	2163	2163
6	64	729	3431	8891	14771	18011	18731
7	128	2187	14325	49731	106851	158931	184131
8	256	6561	59231	272675	757019	1407179	1921259
9	512	19683	242973	1468203	5228043	12200883	20214483

The number $P(n, k)$ of distinct equivalence classes of $(k+1)$-valued assignments over n variables.

Main result

Theorem

Fix integers $n, k \geq 1$, and a formula $\varphi \in \textsc{Form}_n$.

1. $\chi_k(\varphi)$ equals the number of $(k+1)$-valued assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$, up to n-equivalence.

Main result

Theorem

Fix integers $n, k \geq 1$, and a formula $\varphi \in \textsc{Form}_n$.

1. $\chi_k(\varphi)$ equals the number of $(k+1)$-valued assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$, up to n-equivalence.
2. φ is a tautology in $\mathbb{G}_{k+1}$ if and only if $\chi_k(\varphi) = P(n, k)$.

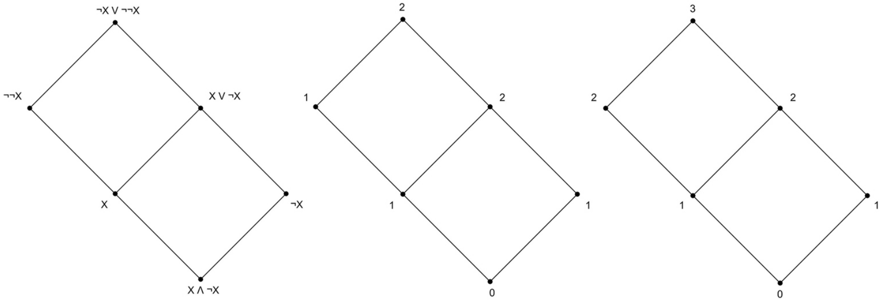
Main result

Theorem

Fix integers $n, k \geq 1$, and a formula $\varphi \in \textsc{Form}_n$.

1. $\chi_k(\varphi)$ equals the number of $(k+1)$-valued assignments $\mu\colon \textsc{Form}_n \to [0,1]$ such that $\mu(\varphi) = 1$, up to n-equivalence.
2. φ is a tautology in $\mathbb{G}_{k+1}$ if and only if $\chi_k(\varphi) = P(n,k)$.
3. φ is a tautology in $\mathbb{G}_\infty$ if and only if it is a tautology in $\mathbb{G}_{n+2}$ if and only if $\chi_{n+1}(\varphi) = P(n, n+1)$.

Example



The Gödel algebra $\mathscr{G}_1$, and the values of $\chi = \chi_1 : \mathscr{G}_1 \longrightarrow \mathbb{R}$ and $\chi_2 : \mathscr{G}_1 \longrightarrow \mathbb{R}$.

Further research

- *Study the logical content of the Euler Characteristic for Łukasiewicz logic, and other many-valued logics.*
- *Find an appropriate version of the Euler characteristic for Łukasiewicz logic, and other many-valued logics.*
- More generally: *Develop the general theory of valuations over Gödel and MV-algebras.*

Thank you for your attention.