A Combinatorial Expansion Arising from Łukasiewicz Logic

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COMBINATORICS 2008

From Łukasiewicz logic to multisets

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- DUAL CATEGORY: multisets with particular morphisms.
- AIM OF THIS TALK: to present a generalization of

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$$

to our multisets.

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- MORPHISMS: A morphism $f : \alpha \to \gamma$, with $\alpha : A \to \mathbb{N}$ and $\gamma : C \to \mathbb{N}$, is a function $f : A \to C$ such that, for all $a \in A$,

  $$\gamma \circ f(a) \mid \alpha(a) \,,$$

  where $s \mid t$ stands for "s divides t".

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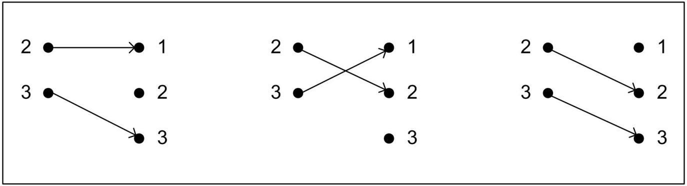
$$\gamma \circ f = \alpha\,.$$
4. We call f *strongly surjective* iff $f \colon A \to C$ is surjective, and for all $c \in C$
$$\gamma(c) = \gcd \{\alpha(a) \mid f(a) = c\}\,.$$

Injections, an example



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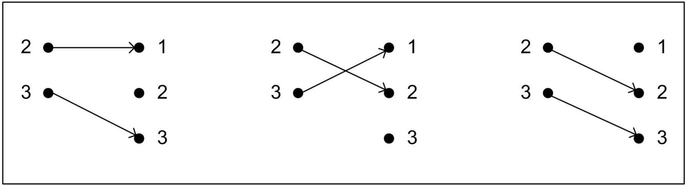

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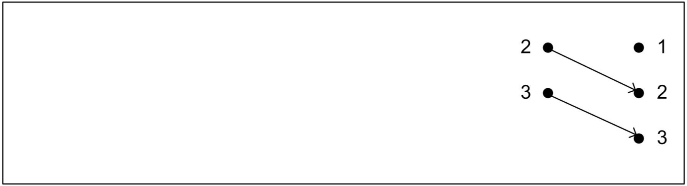

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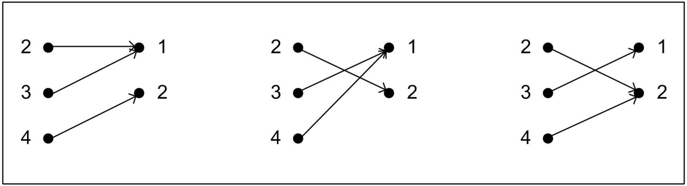

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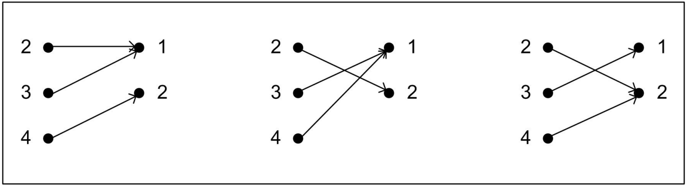

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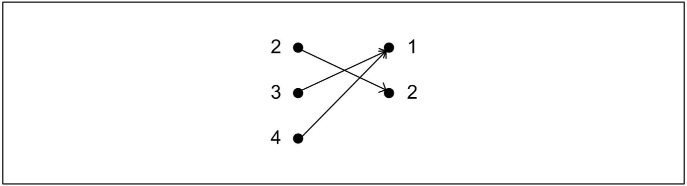

Strong surjections.

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- In our category of multisets, each morphism has a unique weakly surjective-strongly injective factorization, and a unique strongly surjective-weakly injective factorization.
- C_{fin} has no other factorization systems besides the two above.

Partitions of a multiset

- A *weak partition of* α is a multiset π whose underlying elements $\beta_1, \beta_2, ..., \beta_k$ are multisets satisfying the following conditions.
  1. If $B_i = \mathrm{Dom}\,\beta_i$, $i \in \{1, \ldots, k\}$, then $\{B_1, \ldots, B_k\}$ is a partition of A into k blocks.
  2. For every $i \in \{1, \ldots, k\}$, $\pi(\beta_i) \mid \gcd\{\alpha(a) \mid a \in B_i\}$, and $\beta_i(a) = \alpha(a)$ for all $a \in B_i$.

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- EXAMPLE:

$$\pi = \{\{a_1^2, a_4^3\}^1, \{a_3^6, a_2^4\}^2, \{a_5^4\}^4\}$$

is a strong partition of

$$\alpha = \{a_1^2, a_2^4, a_3^6, a_4^3, a_5^4\}$$

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- EXAMPLE:

$$\gamma = \{a_2^8, a_3^6, a_5^4\}$$

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- $((\chi))_\nu$: number of weak injections from a ν-set to a χ-set.

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Theorem

The promised generalizations of $x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$ are obtained as follows.

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For any two partitions of (non-negative) integers ν and χ, we have

$$\sum_{\kappa} \left\{ \left\{ {\nu \atop \kappa} \right\} \right\} (\chi)_\kappa = \chi^\nu = \sum_{\kappa} \left\{ {\nu \atop \kappa} \right\} ((\chi))_\kappa \ ,$$

where κ ranges over all partitions of a (non-negative) integer.

Conclusion

- In this work, we have considered possible generalizations of the basic expansion $x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$ to multisets arising as Birkhoff duals of finite MV algebras.

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- We can think to multisets as swatches of various length, or shelves of various width. By analysing tidy placements of swatches into shelves we shall construct a many-valued analogue of G.C. Rota's Twelvefold Way to Combinatorics.

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- In this framework, some interesting results has been obtained for
  1. Gödel algebras, the algebraic counterpart of Gödel logic, by investigating forests and open maps,
  2. Bounded distributive lattices, by investigating partially ordered sets and order preserving maps.

Thank you

Thank you for your attention ...