Independent Sets of Families of Graphs via Finite State Automata

P. Codara, O.M. D'Antona, M. Galasi, and G. Lavado

Pietro Codara (IIIA-CSIC)

June 30, 2017

The Second Malta Conference in Graph Theory and Combinatorics, 26-30 June 2017, Qawra, Malta

Table of contents

Introduction

Inspiration

An independent set of a graph is a set of pairwise non-adjacent vertices of the graph.

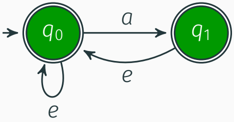
A (two-dimensional) grid graph $G_{m,n}$ is the Cartesian product of the paths P_m and P_n.

The transfer matrix of a grid graph $G_{m,n}$ is a matrix T of size $F_{m+1} \times F_{m+1}$ indexed by the set of independent sets of P_m. We have

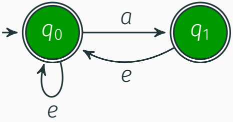
$$T[S_i, S_j] = \begin{cases} 1 & \text{if } S_i \cup S_j \text{ is an independent set of } G_{m,2} \\ 0 & \text{otherwise} \end{cases}$$

The sum of the elements of the matrix $(I - xT)^{-1}$ provides the generating function of the numbers of independent sets of $G_{m,n}$.

Motivation



Motivation

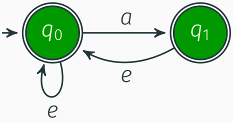


Notation:

n	Independent sets
0	$\varnothing$
1	e, a
2	ee, ea, ae
3	eee, eea, eae, aee, aea

n	Accepted words
0	λ
1	e, a
2	ee, ea, ae
3	eee, eea, eae, aee, aea

Motivation



Notation:

n	Independent sets
0	$\varnothing$
1	e, a
2	ee, ea, ae
3	eee, eea, eae, aee, aea

n	Accepted words
0	λ
1	e, a
2	ee, ea, ae
3	eee, eea, eae, aee, aea

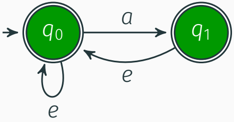
	e	a
e	1	1
a	1	0

[*Calkin&Wilf* '98]

$$\begin{cases} S(x) = & xS(x) + xA(x) + 1 \\ A(x) = & xS(x) + 1 \end{cases}$$

[*Chomsky et al.* '63, *Gruger et al.* '12]

Motivation



Notation:

n	Independent sets
0	$\varnothing$
1	e, a
2	ee, ea, ae
3	eee, eea, eae, aee, aea

n	Accepted words
0	λ
1	e, a
2	ee, ea, ae
3	eee, eea, eae, aee, aea

	e	a
e	1	1
a	1	0

[*Calkin&Wilf* '98]

$$\begin{cases} S(x) = & xS(x) + xA(x) + 1 \\ A(x) = & xS(x) + 1 \end{cases}$$

[*Chomsky et al.* '63, *Gruger et al.* '12]

$$S(x) = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + 13x^5 + \mathcal{O}(x^6)$$

Telescopic families of graphs

Telescopic families of graphs, definition

M, the module, is any graph with $m > 0$ vertices $v_1, v_2, \ldots, v_m$.

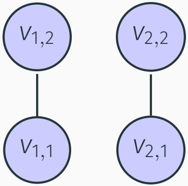
Telescopic families of graphs, definition

Fix a non-negative integer h (the power).

$F_{M,h}$, the h-frame of M, is the graph such that

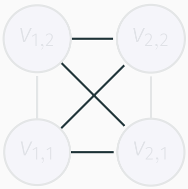
$$V(F_{M,h}) = \bigcup_{i=1}^{h+1} V_i, \text{ where } V_i = \{v_{i,j} \mid j = 1, 2, \ldots, m\}, \text{ and}$$

$$E(F_{M,h}) = \bigcup_{i=1}^{h+1} \{(v_{i,t}, v_{i,t'}) \mid (v_t, v_{t'}) \in E(M)\}.$$



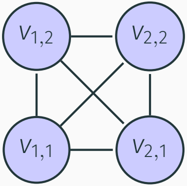
Telescopic families of graphs, definition

X, the cross-connection of $F_{M,h}$, is any subset of $\{(v_{i,j}, v_{i',j'}) \mid 1 \leq i < i' \leq h+1; v_{i,j} \in V_i; v_{i',j'} \in V_{i'}\}$, the set of *inter-layer edges* of $F_{M,h}$.



Telescopic families of graphs, definition

C, the connection, is obtained by adding the inter-layer edges of X to the h-frame of M.



Telescopic families of graphs, definition

Definition

A telescopic family of graphs, TFG, is a sequence of graphs $\{G_n\}_{n\geq 0}$ identified by a triplet (M, h, X).

The graphs of $\{G_n\}_{n\geq 0}$ are:

(i) G_0, that is the empty graph,

(ii) for $1 \leq n \leq h+1$, G_n is the subgraph of the connection C induced by $V_1 \cup V_2 \cup \cdots \cup V_n$,

(iii) for $n > h+1$, G_n is defined by letting

$$V(G_n) = V(G_{n-1}) \cup \{v_{n,j} \mid j = 1, 2, \ldots, m\}, \text{ and}$$
$$E(G_n) = E(G_{n-1}) \cup \{(v_{i+1,j}, v_{i'+1,j'}) \mid (v_{i,j}, v_{i',j'}) \in E(G_{n-1})\}.$$

Telescopic families of graphs, definition

Definition

A telescopic family of graphs, TFG, is a sequence of graphs $\{G_n\}_{n\geq 0}$ identified by a triplet (M, h, X).

The graphs of $\{G_n\}_{n\geq 0}$ are:

(i) G_0, that is the empty graph,

(ii) for $1 \leq n \leq h+1$, G_n is the subgraph of the connection C induced by $V_1 \cup V_2 \cup \cdots \cup V_n$,

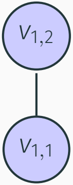
(iii) for $n > h+1$, G_n is defined by letting

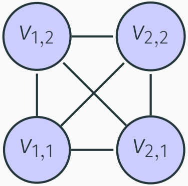
$$V(G_n) = V(G_{n-1}) \cup \{v_{n,j} \mid j = 1, 2, \ldots, m\}, \text{ and}$$
$$E(G_n) = E(G_{n-1}) \cup \{(v_{i+1,j}, v_{i'+1,j'}) \mid (v_{i,j}, v_{i',j'}) \in E(G_{n-1})\}.$$

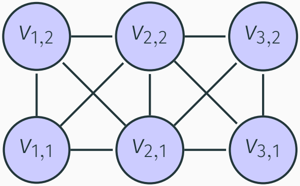
Remarks. (a) $G_1 \simeq M$. (b) $G_{h+1} = C$.

Telescopic families of graphs, examples

The graphs Z_1, Z_2 and Z_3 of the TFG $\{Z_n\}_{n\geq 0}$ identified by (M, h, X), for M, h, X as in the previous examples, are:

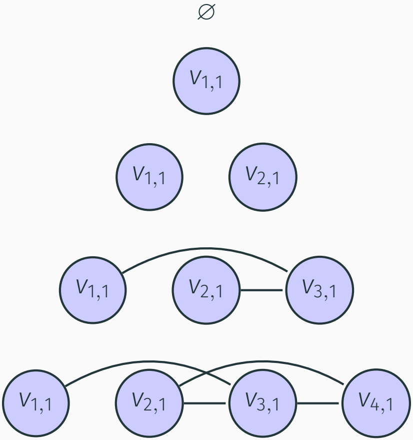






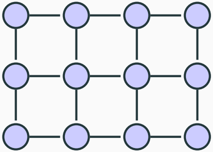
Telescopic families of graphs, examples

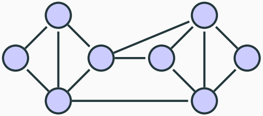
The TFG $\{G_n\}_{n\geq 0}$ identified by $(K_1, 2, \{(v_{1,1}, v_{3,1}), (v_{2,1}, v_{3,1})\})$ contains the following graphs:



Telescopic families of graphs, examples

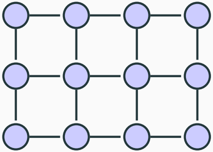
We can construct TFGs of the following forms

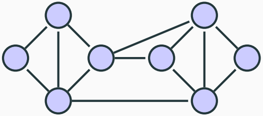




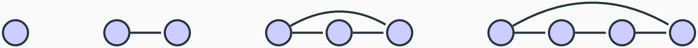
Telescopic families of graphs, examples

We can construct TFGs of the following forms





...but the following family is not a TFG.



The Independence Automaton

Running example

In this part of the presentation we make use of the telescopic family $\{G_n\}_{n \geq 0}$ identified by $(K_1, 2, \{(v_{1,1}, v_{3,1}), (v_{2,1}, v_{3,1})\})$ as a running example.

Running example

In this part of the presentation we make use of the telescopic family $\{G_n\}_{n\geq 0}$ identified by $(K_1, 2, \{(v_{1,1}, v_{3,1}), (v_{2,1}, v_{3,1})\})$ as a running example.

We also make use of a simplified (not general) notation, in order to allow an easy graphical representation of our results.

Running example

In this part of the presentation we make use of the telescopic family $\{G_n\}_{n\geq 0}$ identified by $(K_1, 2, \{(v_{1,1}, v_{3,1}), (v_{2,1}, v_{3,1})\})$ as a running example.

We also make use of a simplified (not general) notation, in order to allow an easy graphical representation of our results.

We now show how to build, in a systematic way, an Independence Automaton (IA) for $\{G_n\}_{n\geq 0}$, that is a deterministic finite automaton $A_{M,h,X} = (\Sigma, Q, q_0, F, \delta)$ that accepts a language in which the number of n-symbol words equals the number of independent sets of G_n, for any $n \geq 0$.

Alphabet

The alphabet is obtained by assigning a symbol to each independent set of the module M (via a bijection ϕ).

Example

- We have two independent sets of the module $M = K_1$. This is ϕ:

$$\varnothing \mapsto e \qquad \{v_1\} \mapsto a$$

- The alphabet of our IA is

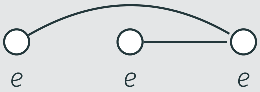
$$\Sigma = \{e, a\}$$

Legal words and states

The legal words are those strings $w \in \Sigma^*$ of length at most $h+1$ which are associated with the independent sets of the graphs $G_0, G_1, \ldots, G_{h+1}$.

Example

- eee, eea are legal words.

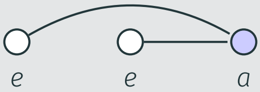


Legal words and states

The legal words are those strings $w \in \Sigma^*$ of length at most $h+1$ which are associated with the independent sets of the graphs $G_0, G_1, \ldots, G_{h+1}$.

Example

- eee, eea are legal words.

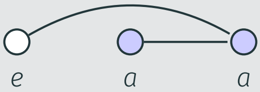


Legal words and states

The legal words are those strings $w \in \Sigma^*$ of length at most $h+1$ which are associated with the independent sets of the graphs $G_0, G_1, \ldots, G_{h+1}$.

Example

- eee, eea are legal words. eaa is not.

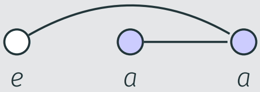


Legal words and states

The legal words are those strings $w \in \Sigma^*$ of length at most $h+1$ which are associated with the independent sets of the graphs $G_0, G_1, \ldots, G_{h+1}$.

Example

- eee, eea are legal words. eaa is not.



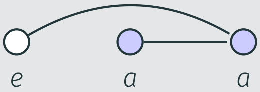
- The set of states of our IA is $Q = \{q_w \mid w \text{ is a legal word}\}$.

Legal words and states

The legal words are those strings $w \in \Sigma^*$ of length at most $h + 1$ which are associated with the independent sets of the graphs $G_0, G_1, \ldots, G_{h+1}$.

Example

- eee, eea are legal words. eaa is not.



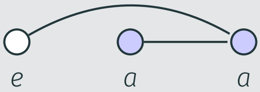
- The set of states of our IA is $Q = \{q_w \mid w \text{ is a legal word}\}$.
- q_λ is the initial state.

Legal words and states

The legal words are those strings $w \in \Sigma^*$ of length at most $h+1$ which are associated with the independent sets of the graphs $G_0, G_1, \ldots, G_{h+1}$.

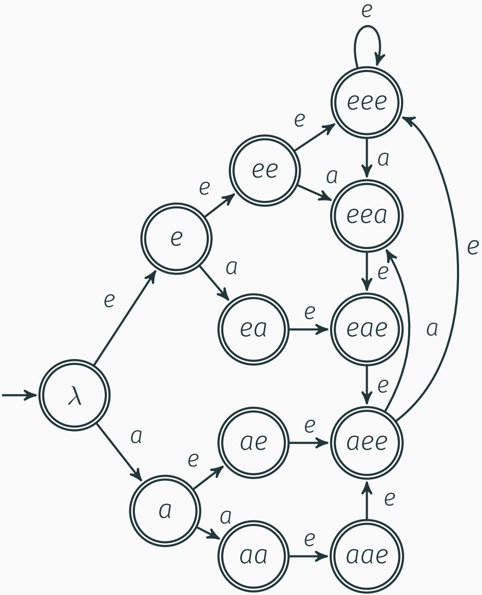
Example

- eee, eea are legal words. eaa is not.



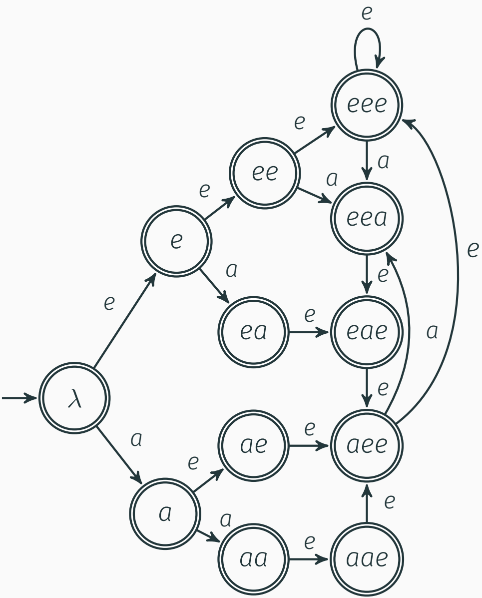
- The set of states of our IA is $Q = \{q_w \mid w \text{ is a legal word}\}$.
- q_λ is the initial state.
- All states are accepting states.

Transitions



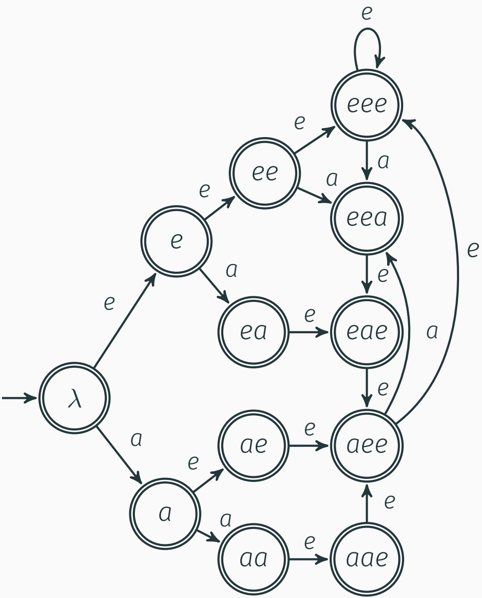
State transition diagram of the IA

Transitions



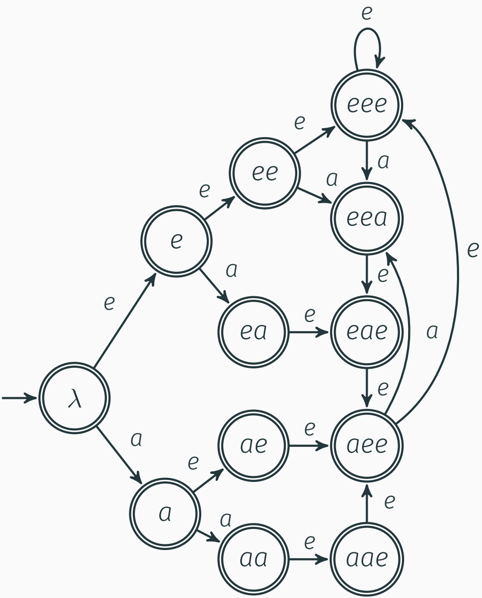
- $\delta(q_{ea}, e) = q_{eae}$

Transitions

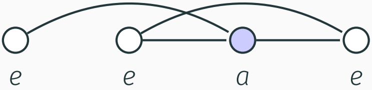


- $\delta(q_{eea}, e) = q_{eae}$

Transitions



- $\delta(q_{eea}, e) = q_{eae}$



Properties of the independence automaton

- Every Independence Automaton has at least $m + 1$ states: q_λ and one for each vertex of M.

Properties of the independence automaton

- Every Independence Automaton has at least $m + 1$ states: q_λ and one for each vertex of M.
- The latter implies that the IA of a TFG is not minimal. That is, one can find another automaton accepting the same language, but having a smaller set of states.

Properties of the independence automaton

- Every Independence Automaton has at least $m+1$ states: q_λ and one for each vertex of M.
- The latter implies that the IA of a TFG is not minimal. That is, one can find another automaton accepting the same language, but having a smaller set of states.
- However, as shown with our example, we can build an Independence Automaton of a TFG in a systematic way.

Properties of the independence automaton

- Every Independence Automaton has at least $m+1$ states: q_λ and one for each vertex of M.
- The latter implies that the IA of a TFG is not minimal. That is, one can find another automaton accepting the same language, but having a smaller set of states.
- However, as shown with our example, we can build an Independence Automaton of a TFG in a systematic way.
- Further, if z is the number of independent sets of the module M, the number of states of $A_{M,h,X}$ is bounded from above by

$$\sum_{t=0}^{h+1} z^t = \frac{z^{h+2}-1}{z-1} \ .$$

Main result and proof

Formalising

Let $\{G_n\}_{n\geq 0}$ be a TFG identified by (M, h, X). Denote by $A_{M,h,X}$ the Independence Automaton of $\{G_n\}_{n\geq 0}$.

- Let $\mathcal{I}$ be the set of all independent sets of M, with $|\mathcal{I}| = z$. We fix a bijection $\phi : \mathcal{I} \to \{0, \ldots, z-1\}$ which assigns a non-negative integer to each independent set of M.

$$\Sigma = \{a_0, \ldots, a_{z-1}\} = \{a_{\phi(Y)} \mid Y \in \mathcal{I}\}.$$

Formalising

Let $\{G_n\}_{n\geq 0}$ be a TFG identified by (M, h, X). Denote by $A_{M,h,X}$ the Independence Automaton of $\{G_n\}_{n\geq 0}$.

- Let $\mathcal{I}$ be the set of all independent sets of M, with $|\mathcal{I}| = z$. We fix a bijection $\phi : \mathcal{I} \to \{0, \ldots, z-1\}$ which assigns a non-negative integer to each independent set of M.

$$\Sigma = \{a_0, \ldots, a_{z-1}\} = \{a_{\phi(Y)} \mid Y \in \mathcal{I}\}.$$

- We define a family of functions $\Psi_n : \Sigma^n \to \mathcal{P}(V(G_n))$ by letting, for each $w = c_1 c_2 \cdots c_n \in \Sigma^n$, $n \geq 1$,

$$\Psi_n(w) = \bigcup_{i=1}^{n} \{v_{i,j_1}, v_{i,j_2}, \ldots, v_{i,j_q} \mid c_i = a_t;\ \phi^{-1}(t) = \{v_{j_1}, v_{j_2}, \ldots, v_{j_q}\}\}.$$

  Further, we let $\Psi_0(\lambda) = \varnothing$.

Formalising

Let $\{G_n\}_{n\geq 0}$ be a TFG identified by (M, h, X). Denote by $A_{M,h,X}$ the Independence Automaton of $\{G_n\}_{n\geq 0}$.

- Let $\mathcal{I}$ be the set of all independent sets of M, with $|\mathcal{I}| = z$. We fix a bijection $\phi : \mathcal{I} \to \{0, \ldots, z-1\}$ which assigns a non-negative integer to each independent set of M.

$$\Sigma = \{a_0, \ldots, a_{z-1}\} = \{a_{\phi(Y)} \mid Y \in \mathcal{I}\}.$$

- We define a family of functions $\Psi_n : \Sigma^n \to \mathcal{P}(V(G_n))$ by letting, for each $w = c_1 c_2 \cdots c_n \in \Sigma^n$, $n \geq 1$,

$$\Psi_n(w) = \bigcup_{i=1}^{n} \{v_{i,j_1}, v_{i,j_2}, \ldots, v_{i,j_q} \mid c_i = a_t;\ \phi^{-1}(t) = \{v_{j_1}, v_{j_2}, \ldots, v_{j_q}\}\}.$$

  Further, we let $\Psi_0(\lambda) = \varnothing$.
- We say that $w \in \Sigma^*$ is a legal word if $|w| \leq h+1$, and for each $(v_{i,j}, v_{i',j'}) \in X$, $\{v_{i,j}, v_{i',j'}\} \not\subseteq \Psi_{|w|}(w)$.

Main result

- The set of states of $A_{M,h,X}$ is $Q(A_{M,h,X}) = \{q_w \mid w \text{ is a legal word}\}$.

Main result

- The set of states of $A_{M,h,X}$ is $Q(A_{M,h,X}) = \{q_w \mid \; w \text{ is a legal word}\}$.
- We define a partition of $Q(A_{M,h,X})$ in the following way.

  For $i = 0, \ldots, h+1,\; Q_i = \{q_w \mid w \text{ is a legal word of length } i\}$.

  This allows us to see $Q(A_{M,h,X})$ as a layered structure.
- We define the transitions of $A_{M,h,X}$, as follows.
  1. For $0 \leq i \leq h$, $q_w \in Q_i$, $a_j \in \Sigma$, we set $\delta(q_w, a_j) = q_{wa_j}$ iff $q_{wa_j} \in Q_{i+1}$.
  2. For $a_k \in \Sigma$, $\bar{w} \in \Sigma^*$, $|\bar{w}| = h$, $w = a_k\bar{w}$, $q_w \in Q_{h+1}$, and $a_j \in \Sigma$, we set $\delta(q_w, a_j) = q_{\bar{w}a_j}$ iff $q_{\bar{w}a_j} \in Q_{h+1}$.

Main result

- The set of states of $A_{M,h,X}$ is $Q(A_{M,h,X}) = \{q_w \mid w \text{ is a legal word}\}$.
- We define a partition of $Q(A_{M,h,X})$ in the following way.

$$\text{For } i = 0, \ldots, h+1,\ Q_i = \{q_w \mid w \text{ is a legal word of length } i\}.$$

  This allows us to see $Q(A_{M,h,X})$ as a layered structure.
- We define the transitions of $A_{M,h,X}$, as follows.
  1. For $0 \leq i \leq h$, $q_w \in Q_i$, $a_j \in \Sigma$, we set $\delta(q_w, a_j) = q_{wa_j}$ iff $q_{wa_j} \in Q_{i+1}$.
  2. For $a_k \in \Sigma$, $\bar{w} \in \Sigma^*$, $|\bar{w}| = h$, $w = a_k\bar{w}$, $q_w \in Q_{h+1}$, and $a_j \in \Sigma$, we set $\delta(q_w, a_j) = q_{\bar{w}a_j}$ iff $q_{\bar{w}a_j} \in Q_{h+1}$.

Theorem

Let $w \in \Sigma^*$ and $A_{M,h,X} = (\Sigma, Q, q_\lambda, F, \delta)$ be the Independence Automaton of a TFG *$\{G_n\}_{n\geq 0}$ identified by (M, h, X). Then, $w \in L(A_{M,h,X})$ if and only if $\Psi_{|w|}(w)$ is an independent set of $G_{|w|}$.*

Sketch of the proof

1. w is a legal word iff $\Psi_{|w|}(w)$ is an independent set of $G_{|w|}$.
2. $w \in L(A_{M,h,X})$ iff either $|w| < h+1$ and w is a legal word, or $|w| \geq h+1$ and each factor of w of length $h+1$ is a legal word.

This concludes the proof of the case $|w| \leq h+1$.

3. Let $e = (v_{i,j}, v_{i',j'})$ be any edge of G_n. Then $|i' - i| < h+1$.
4. For all $t > 0$, for all $n \geq h+t$, if $e = (v_{i,j}, v_{i',j'}) \in X$, then $(v_{i+t-1,j}, v_{i'+t-1,j'}) \in E(G_n)$.
5. For $t > 0$, and $e = (v_{i,j}, v_{i',j'}) \in X$, let $\tau_t(e) = \{v_{i+t-1,j}, v_{i'+t-1,j'}\}$. Let $|w| = l \geq h+1$. Then, $\Psi_l(w)$ is an independent set of G_l if and only if for all $t \in \{1, \ldots, l-h\}$ and for each $e \in X$, $\tau_t(e) \not\subseteq \Psi_l(w)$.
6. Let $|w| = l \geq h+1$. Then, $\Psi_l(w)$ is an independent set of G_l iff for each factor $\bar{w}$ of w having length $h+1$, $\Psi_{h+1}(\bar{w})$ is an independent set of G_{h+1}.

This concludes the proof of the case $|w| > h+1$.

Further work

The cyclic expansion of a TFG

- Following this work, we have been working on an extension of our class of families of graphs on which a similar technique can be adopted. The main aim was to allow dealing with cycles.

The cyclic expansion of a TFG

- Following this work, we have been working on an extension of our class of families of graphs on which a similar technique can be adopted. The main aim was to allow dealing with cycles.
- In fact, we can produce a cyclic expansion of each TFG, and construct the Independence Automaton of the new family.

The cyclic expansion of a TFG

- Following this work, we have been working on an extension of our class of families of graphs on which a similar technique can be adopted. The main aim was to allow dealing with cycles.
- In fact, we can produce a cyclic expansion of each TFG, and construct the Independence Automaton of the new family.
- The “cyclic expansion” of the Independence Automaton turns out to be, in general, larger and more complex than the previuos one. In particular, it contains both final and non-final states.

References

References

N. J. Calkin and H. S. Wilf.
The Number of Independent Sets in a Grid Graph.
SIAM Journal on Discrete Mathematics, 11(1):54–60, 1998.

N. Chomsky and M. Schützenberger.
The Algebraic Theory of Context-Free Languages.
In *Studies in Logic and the Foundations of Mathematics*, volume 35, pages 118–161. Elsevier, 1963.

H. Gruber, J. Lee, and J. Shallit.
Enumerating regular expressions and their languages.
CoRR, abs/1204.4982, 2012.

Thank you for your attention.