On Gödel Algebras of Concepts

Pietro Codara[1] and Diego Valota[2]

[1] Artificial Intelligence Research Institute (IIIA-CSIC),
Campus de la UAB, 08193 Bellaterra, Spain
`codara@di.unimi.it`

[2] Dipartimento di Informatica, Università degli Studi di Milano,
Via Comelico 39, 20135 Milano, Italy
`diego@iiia.csic.es`

Abstract. Beside algebraic and proof-theoretical studies, a number of different approaches have been pursued in order to provide a complete intuitive semantics for many-valued logics. Our intention is to use the powerful tools offered by *formal concept analysis* (FCA) to obtain further intuition about the intended semantics of a prominent many-valued logic, namely Gödel, or Gödel-Dummett, logic. In this work, we take a first step in this direction. Gödel logic seems particularly suited to the approach we aim to follow, thanks to the properties of its corresponding algebraic variety, the class of *Gödel algebras*. Furthermore, Gödel algebras are prelinear Heyting algebras. This makes Gödel logic an ideal contact-point between intuitionistic and many-valued logics.

In the literature one can find several studies on relations between FCA and fuzzy logics. These approaches often amount to equipping both intent and extent of concepts with connectives taken by some many-valued logic. Our approach is different. Since Gödel algebras are (residuated) lattices, we want to understand which type of concepts are expressed by these lattices. To this end, we investigate the concept lattice of the standard context obtained from the lattice reduct of a Gödel algebra. We provide a characterization of Gödel implication between concepts, and of the Gödel negation of a concept. Further, we characterize a Gödel algebra of concepts. Some concluding remarks will show how to associate (equivalence classes of) formulæ of Gödel logic with their corresponding formal concepts.

Keywords: Intended Semantics, Concept Lattice, Many-Valued Logic, FCA, Formal Concept Analysis, Fuzzy Logic, Gödel Logic

1 Introduction

Gödel logic can be semantically defined as a many-valued logic, as follows. Consider the set FORM of well-formed formulæ over propositional variables $\{x_1, x_2, x_3, \dots\}$ in the language $(\wedge, \vee, \to, \bot, \top)$. An *assignment* is a function μ

from FORM to $[0,1] \subseteq \mathbb{R}$, such that, for any $\varphi, \psi \in$ FORM,

$$\mu(\bot) = 0\,, \qquad \mu(\top) = 1\,,$$
$$\mu(\varphi \wedge \psi) = \min\{\mu(\varphi), \mu(\psi)\}\,,$$
$$\mu(\varphi \vee \psi) = \max\{\mu(\varphi), \mu(\psi)\}\,,$$
$$\mu(\varphi \to \psi) = \begin{cases} 1 & \text{if } \mu(\varphi) \leq \mu(\psi)\,, \\ \mu(\psi) & \text{otherwise}\,. \end{cases}$$

A formula φ such that $\mu(\varphi) = 1$ for every assignment μ is called a *tautology*. To indicate such a case we write $\vDash \varphi$.

Gödel logic can also be syntactically defined as a schematic extension of intuitionistic propositional calculus by the *prelinearity axiom*

$$(\varphi \to \psi) \vee (\psi \to \varphi). \tag{P}$$

We write $\vdash \varphi$ to mean that the formula φ is derivable from the axioms of Gödel logic using *modus ponens* as the only deduction rule. Gödel logic is complete with respect to the many-valued semantics defined above: in symbols, $\vdash \varphi$ if and only if $\vDash \varphi$. Details and proofs can be found in [22].

Even though Gödel logic is an axiomatic extension of intuitionistic logic, the constructive intended semantics[3] of the latter is not suitable for the former. Indeed, think of formulæ of FORM as problems for which we have an algorithmic solution. Then, (P) states that, for every choice of φ and ψ in FORM, the solution to φ can be reduced to the solution to ψ, or the solution to ψ can be reduced to the solution to φ. A rather strong assumption. This is a common problem of informal intended semantics. They are tailored over a specific logic. Applying them to some extension is not straightforward, or not even possible.

On the other hand, beside algebraic and proof-theoretical studies, a number of different approaches have been attempted to provide semantics for Gödel logics. To mention a few, we cite [5] and [18], where temporal-like and game-theoretic semantics, respectively, are investigated.

The possibility of connecting descriptions of real-world contexts with powerful formal instruments is what makes *formal concept analysis* (FCA) a promising framework, merging the intuitions of intended semantics with the advantages of formal semantics. In the present work, we study formal contexts associated with Gödel logic from the algebraic point of view. The algebraic semantics of Gödel logic is the subvariety of Heyting algebras satisfying prelinearity. A *Heyting algebra* is a structure $\mathbf{A} = (A, \wedge, \vee, \to, \top, \bot)$ of type $(2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, \top, \bot)$ is a distributive lattice and the couple $(\wedge, \to)$ forms a *residuated pair*. This means that the unique operation $\to$ that satisfies the residuation property, $x \wedge z \leq y$ if and only if $z \leq x \to y$, is the *residuum* of $\wedge$, defined as

$$x \to y = \max\{z \mid x \wedge z \leq y\}. \tag{1}$$

[3] The intended semantics of a logical language consists of the collection of models that intuitively the language talks about. In this specific case the intended semantics' is the informal description of *truth as provability* given by Brouwer.

Hence, a *Gödel algebra* is a Heyting algebra satisfying the prelinearity equation $(x \to y) \vee (y \to x) = \top$, for $x, y \in A$. Horn [23] showed that the variety of Gödel algebras is *locally finite*. That is, the classes of finite, finitely generated and finitely presented algebras coincide.

For an integer $n \geq 1$, let FORM_n be the set of all formulæ whose propositional variables are contained in $\{x_1, \ldots, x_n\}$. Two formulæ $\varphi, \psi \in \mathrm{FORM}_n$ are called *logically equivalent* if both $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$ hold. Logical equivalence is an equivalence relation, denoted by $\equiv$. We denote the equivalence class of a formula φ by $[\varphi]_\equiv$. It is straightforward to see that the quotient set $\mathrm{FORM}_n/\equiv$, endowed with the operations $\wedge, \vee, \top, \bot$ induced by the corresponding logical connectives, is a distributive lattice with top and bottom element $\top$ and $\bot$, respectively. If, in addition, $\mathrm{FORM}_n/\equiv$ is endowed with the operation $\to$ induced by the logical implication, then $\mathrm{FORM}_n/\equiv$ becomes a Gödel algebra. The specific Gödel algebra $\mathcal{G}_n = \mathrm{FORM}_n/\equiv$ is, by construction, the *Lindenbaum algebra* of Gödel logic over the language $\{x_1, \ldots, x_n\}$. Lindenbaum algebras are isomorphic to free algebras, thus $\mathcal{G}_n$ is the free n-generated Gödel algebra. Moreover, since the variety of Gödel algebras is locally finite, every finite Gödel algebra can be obtained as a quotient of a free n-generated Gödel algebra. For the rest of this paper, all Gödel algebras are assumed to be finite.

In the next Section, we recall some basic notions on FCA. In Section 3 we deal with the concept lattice $\mathbf{C_A}$ of the standard context obtained from a Gödel algebra $\mathbf{A}$. We prove that endowing $\mathbf{C_A}$ with a suitable implication between concepts, we obtain an algebra of concepts isomorphic to $\mathbf{A}$. Further, we characterize the Gödel negation in terms of concepts. In Section 4 we characterize Gödel algebras of concepts. In Section 5 we show how to associate concepts belonging to a Gödel algebras of concepts with Gödel logic formulæ. Finally, in Section 6 we discuss the integration of this approach with the studies on many-valued (substructural) logics aimed to investigate their intended semantics.

2 Basic Notions on FCA

We recollect the basic definitions and facts about formal concept analysis needed in this work. For further details on this topics we refer the reader to [20].

Recall that an element j of a distributive lattice L is called a *join-irreducible* if j is not the bottom of L and if whenever $j = a \vee b$, then $j = a$ or $j = b$, for $a, b \in L$. Meet-irreducible elements are defined dually. Given a lattice $L = (L, \sqcap, \sqcup, 1)$, we denote by $\mathfrak{J}(L)$ the set of its join-irreducible elements, and by $\mathfrak{M}(L)$ the set of its meet-irreducible elements.

Let G and M be arbitrary sets of *objects* and *attributes*, respectively, and let $I \subseteq G \times M$ be an arbitrary binary relation. Then, the triple $\mathbb{K} = (G, M, I)$ is called a *formal context*. For $g \in G$ and $m \in M$, we interpret $(g, m) \in I$ as "the object g has attribute m". For $A \subseteq G$ and $B \subseteq M$, a Galois connection between the powersets of G and M is defined through the following operators:

$$A' = \{m \in M \mid \forall g \in A : gIm\} \qquad B' = \{g \in G \mid \forall m \in B : gIm\}$$

Every pair (A, B) such that $A' = B$ and $B' = A$ is called a *formal concept.* A and B are the *extent* and the *intent* of the concept, respectively. Given a context $\mathbb{K}$, the set $\mathfrak{B}(\mathbb{K})$ of all formal concepts of $\mathbb{K}$ is partially ordered by $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ (or, equivalently, $B_2 \subseteq B_1$). The *basic theorem on concept lattices* [20, Theorem 3] states that the set of formal concepts of the context $\mathbb{K}$ is a complete lattice $(\mathfrak{B}(\mathbb{K}), \sqcap, \sqcup)$, called *concept lattice*, where meet and join are defined by:

$$\begin{aligned} \bigsqcap_{j\in J}(A_j, B_j) &= \left(\bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)''\right) , \\ \bigsqcup_{j\in J}(A_j, B_j) &= \left(\left(\bigcup_{j\in J} A_j\right)'', \bigcap_{j\in J} B_j\right) , \end{aligned} \tag{2}$$

for a set J of indexes. The following proposition is fundamental for our treatise.

Proposition 1 ([20, Proposition 12]). *For every finite lattice L there is (up to isomorphisms) a unique context $\mathbb{K}_L$, with $L \cong \mathfrak{B}(\mathbb{K}_L)$:*

$$\mathbb{K}_L := (\mathfrak{J}(L), \mathfrak{M}(L), \leq).$$

The context $\mathbb{K}_L$ is called the *standard context* of the lattice L.

Since L is finite, $\mathfrak{J}(L)$ is finite. Hence, the concept $(\mathfrak{J}(L), \emptyset)$ is the top element of $\mathfrak{B}(\mathbb{K}_L)$. We denote it $\top_G$, emphasizing the fact that the join-irreducible elements of L are the objects of our context. Analogously, the concept $(\emptyset, \mathfrak{M}(L))$ is the bottom element of $\mathfrak{B}(\mathbb{K}_L)$, and we denote it by $\bot_M$.

Example 1. Let $L = (\{a, b, c, d, e, f\}, \leq)$ be the finite distributive lattice in Figure 1(a). Then, $\mathfrak{J}(L) = \{b, c, e\}$, and $\mathfrak{M}(L) = \{b, d, e\}$. Let $G = \{g_1, g_2, g_3\}$, and $M = \{m_1, m_2, m_3\}$. We relabel $\mathfrak{J}(L)$, and $\mathfrak{M}(L)$ via the labeling functions $\lambda_J : \mathfrak{J}(L) \to G$, and $\lambda_M : \mathfrak{M}(L) \to M$ such that $\lambda_J(b) = g_1$, $\lambda_J(c) = g_2$, $\lambda_J(e) = g_3$, $\lambda_M(b) = m_1$, $\lambda_M(d) = m_2$, and $\lambda_M(e) = m_3$. The following tables show the standard context $\mathbb{K}_L$, and its relabeling in terms of G and M:

$\leq$	b	d	e
b	×	×	
c		×	×
e			×

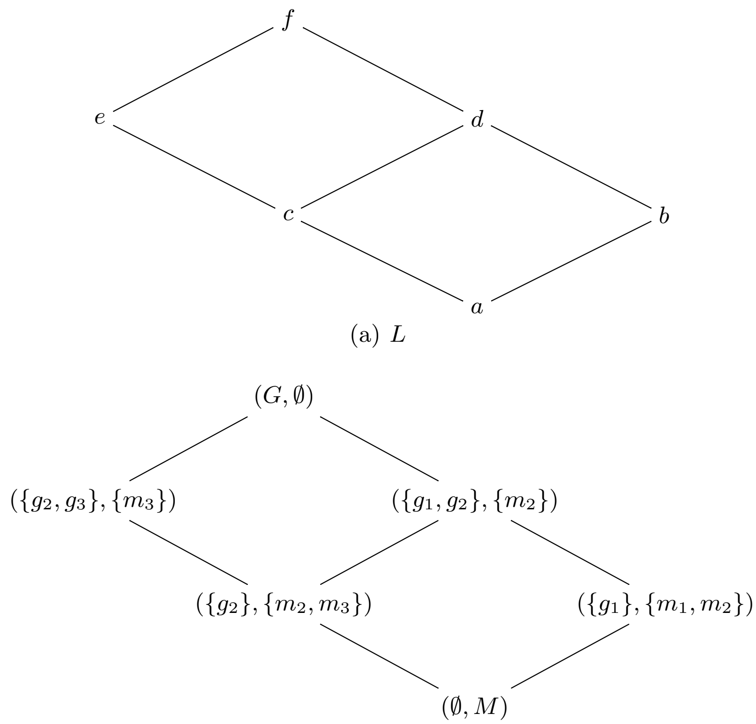
$\leq$	m_1	m_2	m_3
g_1	×	×	
g_2		×	×
g_3			×

The concept lattice $\mathfrak{B}(\mathbb{K}_L)$ is depicted in Figure 1(b).

3 Gödel Algebras of Concepts

Definition 1. *Let $\mathbb{K}$ be a finite context, and let $\mathfrak{B}(\mathbb{K})$ be its concept lattice. For every two concepts $C_1 = (G_1, M_1)$ and $C_2 = (G_2, M_2)$ in $\mathfrak{B}(\mathbb{K})$, we define the* p-implication $(\Rightarrow)$ *as:*

$$C_1 \Rightarrow C_2 = \bigsqcup \{(G_k, M_k) \in \mathfrak{B}(\mathbb{K}) \mid M_k \supseteq M_2 \setminus M_1\} . \tag{$\Rightarrow$}$$



(a) L

(b) $\mathfrak{B}(\mathbb{K}_L)$

Fig. 1: A finite distributive lattice L, and its corresponding concept lattice $\mathfrak{B}(\mathbb{K}_L)$.

The following example better clarifies the previous definition.

Example 2. Consider the concept lattice depicted in Figure 1(b). Then,

$$(\{g_1, g_2\}, \{m_2\}) \Rightarrow (\{g_2\}, \{m_2, m_3\}) = (\{g_2, g_3\}, \{m_3\}) \,,$$
$$(\{g_2\}, \{m_2, m_3\}) \Rightarrow (\emptyset, M) = (\{g_1\}, \{m_1, m_2\}) \,.$$

The following proposition provides a way to build a concept lattice isomorphic to every Gödel algebra.

Proposition 2. *Let* $\mathbf{A} = (A, \wedge, \vee, \rightarrow, \top, \bot)$ *be a Gödel algebra, and let* $C_{\mathbf{A}} = \mathfrak{B}((\mathfrak{J}(\mathbf{A}), \mathfrak{M}(\mathbf{A}), \leq))$ *be the concept lattice of its standard context. Then, the algebra* $\mathbf{C_A} = (C_{\mathbf{A}}, \sqcap, \sqcup, \Rightarrow, \top_G, \bot_M)$*, where* $\Rightarrow$ *is the p-implication, is isomorphic to* $\mathbf{A}$.

Proof. Since each Gödel algebra is a finite lattice, it is isomorphic to the concept lattice of the associated standard context (c.f. Proposition 1). Let $f : A \to C_{\mathbf{A}}$

be such an isomorphism. We have to show that f extends to an isomorphism of Gödel algebras, that is

$$f(x \to y) = f(x) \Rightarrow f(y)\,, \tag{3}$$

for each $x, y \in A$. To this end, it suffices to prove the following claim.

Claim. The couple $(\sqcap, \Rightarrow)$ is a residuated pair.

We need to show that $(\sqcap, \Rightarrow)$ satisfies the residuum equation (1). That is

$$(C_1 \Rightarrow C_2) = \bigsqcup \{C_i \in C_{\mathbf{A}} \mid C_i \sqcap C_1 \leq C_2\}\,, \tag{4}$$

for every $C_1 = (G_1, M_1)$ and $C_2 = (G_2, M_2)$ in $\mathbf{C_A}$. We call $C_z = (G_z, M_z) = \bigsqcup \{C_i \in C_{\mathbf{A}} \mid C_i \sqcap C_1 \leq C_2\}$. By Definition 1, we have:

$$(C_1 \Rightarrow C_2) = \bigsqcup \{(G_i, M_i) \in C_{\mathbf{A}} \mid M_i \supseteq M_2 \setminus M_1\} = C_s = (G_s, M_s)\,. \tag{5}$$

We have to show that $M_s = M_z$ (equivalently, $G_s = G_z$). By (4), M_z is the smallest subset of M that belongs to a concept, and such that $M_z \cup M_1 \supseteq M_2$. In other words, M_z is precisely the smallest M_t such that $M_t \supseteq M_2 \setminus M_1$. Hence, by (5), M_z coincides with M_s. This settles the claim.

By the preceding claim, $\Rightarrow$ is precisely the unique (Gödel) residuum of $\sqcap$. Since the lattice isomorphisms f also preserves $\sqcap$, we have shown (3), and our statement is proved. □

We have derived the natural notion of implication between concepts in case the concept lattice is a Gödel algebra. Indeed, the p-implication satisfy the residuation law. It is now easy to provide a characterization of the Gödel negation of a concept.

Definition 2. *Let $\mathfrak{B}(\mathbb{K})$ be a concept lattice over a context $\mathbb{K}$, and let $(G_1, M_1) \in \mathfrak{B}(\mathbb{K})$. We call the* p-complement *of (G_1, M_1) the following operation*:

$$\sim (G_1, M_1) = \bigsqcup \{(G_k, M_k) \in \mathfrak{B}(\mathbb{K}) \mid M_k \supseteq M \setminus M_1\}\,.$$

Corollary 1. *The p-complement is the Gödel negation in a Gödel algebra of concepts.*

Proof. In Gödel logic the negation connective is derived from the implication: $\neg x := x \to \bot$. An easy computation shows that, if C is a concept of a Gödel algebra of concepts, then $\sim C = C \Rightarrow \bot$. □

Example 3. Consider the concept lattice depicted in Figure 1(b). Then,

$$\sim (\{g_1, g_2\}, \{m_2\}) = (\emptyset, M)\,,$$
$$\sim (\{g_2\}, \{m_2, m_3\}) = (\{g_1\}, \{m_1, m_2\}).$$

Compare the second negation with Example 2.

4 Characterizing Gödel Algebras of Concepts

Let $\mathbb{K}$ be a finite context, and let $(\mathfrak{B}(\mathbb{K}), \sqcap, \sqcup)$ be its concept lattice. If, for each $C_1, C_2 \in \mathfrak{B}(\mathbb{K})$, there exists a greatest context $C \in \mathfrak{B}(\mathbb{K})$ such that $C_1 \sqcap C \leq C_2$, then $\mathfrak{B}(\mathbb{K})$ is a residuated lattice. The concept C is called the residuum, and it is denoted by $C_1 \Rightarrow C_2$. Since the residuum, if it exists, is unique, we have that $\Rightarrow$ must be exactly the p-implication defined in Definition 1. Indeed, in the proof of Proposition 2 it is shown that $(\sqcap, \Rightarrow)$ is a residuated pair. In general, a concept lattice need not be a distributive lattice. However, the existence of a residuum respect to the $\sqcap$ implies distributivity. Hence, in order to provide a characterization of Gödel algebras of concepts, we do not need to characterize distributivity. Nonetheless, the characterization of distributivity in concept lattices is an important topic in itself. An intrinsic characterization of distributivity in the finite case is provided in [26]. The infinite case has also been investigated, see [15].

The following proposition characterizes those concept lattices which are Gödel algebras.

Proposition 3. *Let $\mathbb{K}$ be a finite context, and let $(\mathfrak{B}(\mathbb{K}), \sqcap, \sqcup)$ be its concept lattice. Then,*

(i) $(\mathfrak{B}(\mathbb{K}), \sqcap, \sqcup, \Rightarrow, \top_G, \bot_M)$ *is a Heyting algebra if and only if for each $C_1 = (G_1, M_1), C_2 = (G_2, M_2) \in \mathfrak{B}(\mathbb{K})$ there exists a greatest contest $C \in \mathfrak{B}(\mathbb{K})$ such that $C_1 \sqcap C \leq C_2$.*

Moreover, let $C_l = (G_l, M_l) \in \mathfrak{B}(\mathbb{K})$ be such that M_l is the smallest set of attributes satisfying $M_l \supseteq M_2 \setminus M_1$. Analogously, let $C_r = (G_r, M_r) \in \mathfrak{B}(\mathbb{K})$ be such that M_r is the smallest set of attributes satisfying $M_r \supseteq M_1 \setminus M_2$.

(ii) *The Heyting algebra $(\mathfrak{B}(\mathbb{K}), \sqcap, \sqcup, \Rightarrow, \top_G, \bot_M)$ is a Gödel algebra if and only if $M_l \cap M_r = \emptyset$.*

Proof. The first part of the proposition is an immediate translation of the residuation property in terms of concepts. It has already been discussed in the beginning of the present section. We just need to prove (ii). Recall that Gödel algebras are Heyting algebras with a prelinear implication. We have to prove that the p-implication $\Rightarrow$ satisfies the prelinearity equation $(C_1 \Rightarrow C_2) \sqcup (C_2 \Rightarrow C_1) = \top_G$, for every $C_1, C_2 \in \mathfrak{B}(\mathbb{K})$, if, and only if, $M_l \cap M_r = \emptyset$.

Let

$$C_1 \Rightarrow C_2 = C_s = (G_s, M_s) = \bigsqcup \{(G_i, M_i) \in \mathfrak{B}(\mathbb{K}) \mid M_i \supseteq M_2 \setminus M_1\}\,,$$
$$C_2 \Rightarrow C_1 = C_z = (G_z, M_z) = \bigsqcup \{(G_i, M_i) \in \mathfrak{B}(\mathbb{K}) \mid M_i \supseteq M_1 \setminus M_2\}\,.$$

Hence, prelinearity equation can be rewritten as:

$$C_s \sqcup C_z = (\mathfrak{J}(\mathfrak{B}(\mathbb{K})), \emptyset)\,.$$

We observe that $M_l = M_s$, and $M_r = M_z$. Thus, $C_s \sqcup C_z = (\mathfrak{J}(\mathfrak{B}(\mathbb{K})), \emptyset)$ is equivalent to $M_l \cap M_r = \emptyset$, and (ii) is proved. □

5 Formal Concepts Described by Gödel Logic Sentences

In Section 3 we have associated formal concepts with elements of a finite Gödel algebra. Moreover, we have endowed the concept lattice with suitable operations, showing that every Gödel algebra is isomorphic to its associated concept lattice endowed with a p-implication. In this section, we advance some remarks on the logical counterpart of Gödel algebras, namely Gödel logic. Consider the free n-generated Gödel algebra $\mathcal{G}_n$. Since every finite Gödel algebra can be obtained as a quotient of a free n-generated Gödel algebra, we can effectively associate every Gödel logic formula with a corresponding concept. Knowing that $\mathcal{G}_n$ is a finite (distributive) lattice whose elements are formulæ in n variables (up to logical equivalence), and since for every finite lattice there is a unique reduced context $\mathbb{K}$, one can, indeed, relate (equivalence classes of) logical formulæ in $\mathcal{G}_n$ with the concepts in $\mathbb{K}$. That is precisely what we do in this section.

We start with a small example that can be dealt with via a trivial computation: the free 1-generated Gödel algebra $\mathcal{G}_1$. Comparing Figure 2 and Figure 1, one immediately notes that the lattice structure of $\mathcal{G}_1$ is isomorphic to $\mathfrak{B}(\mathbb{K}_L)$ in Figure 1(b). Hence, by Proposition 1, there exists a lattice isomorphism $f : L(\mathcal{G}_1) \to \mathfrak{B}(\mathbb{K}_L)$ such that

$$
\begin{aligned}
&f(\top) = (G, \emptyset)\,, && f(\neg\neg x) = (\{g_2, g_3\}, \{m_3\})\,,\\
&f(x \wedge \neg x) = (\{g_1, g_2\}, \{m_2\})\,, && f(x) = (\{g_2\}, \{m_2, m_3\})\,,\\
&f(\neg x) = (\{g_1\}, \{m_1, m_2\})\,, && f(\bot) = (\emptyset, M)\,.
\end{aligned}
$$

Moreover, by Proposition 2, $\mathfrak{B}(\mathbb{K}_L) = \mathbf{C}_{\mathcal{G}_\mathbf{1}}$ and f is an isomorphism of algebras. Then,

$$
\begin{aligned}
&f([x \vee \neg x]_\equiv \to [x]_\equiv) = f([\neg\neg x]_\equiv) =\\
&= (\{g_2, g_3\}, \{m_3\}) = (\{g_1, g_2\}, \{m_2\}) \Rightarrow (\{g_2\}, \{m_2, m_3\})\,,
\end{aligned}
$$

$$
\begin{aligned}
&f([x]_\equiv \to [\bot]_\equiv) = f([\neg x]_\equiv) =\\
&= (\{g_1\}, \{m_1, m_2\}) = (\{g_2\}, \{m_2, m_3\}) \Rightarrow (\emptyset, M)\,.
\end{aligned}
$$

Compare with Example 2.

Let us consider a more complicated structure. Take the formula $\psi = \neg\neg x_1 \wedge \neg\neg x_2 \wedge (x_1 \vee x_2)$ over $\{x_1, x_2\}$, and let $\mathbf{A}$ be the Gödel algebra $\mathcal{G}_2/(\psi = \top)$ depicted in Figure 3 (note that the equivalence classes displayed are the ones of $\mathcal{G}_2/(\psi = \top)$, not of $\mathcal{G}_2$).

Observe that $\mathfrak{J}(\mathbf{A}) = \{[x_1]_\equiv, [x_2]_\equiv, [x_1 \wedge x_2]_\equiv\}$, and $\mathfrak{M}(\mathbf{A}) = \{[x_1]_\equiv, [x_2]_\equiv\}$. Let $G = \{g_1, g_2, g_3\}$, and $M = \{m_1, m_2\}$, and define the labeling functions $\lambda_J : \mathfrak{J}(L) \to G$ and $\lambda_M : \mathfrak{M}(L) \to M$ by $\lambda_J([x_1 \wedge x_2]_\equiv) = g_1$, $\lambda_J([x_1]_\equiv) = g_2$, $\lambda_J([x_2]_\equiv) = g_3$, $\lambda_M([x_1]_\equiv) = m_1$, and $\lambda_M([x_1]_\equiv) = m_2$. The following two tables provide the standard context $C_\mathbf{A}$, and its relabeling in terms of G and M.

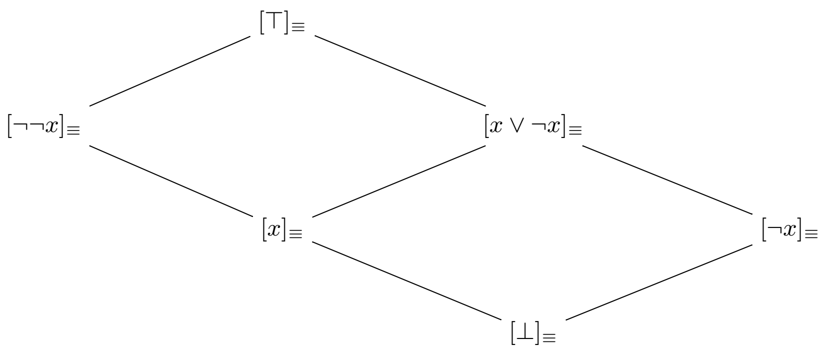


Fig. 2: The free 1-generated Gödel algebra $\mathcal{G}_1$.

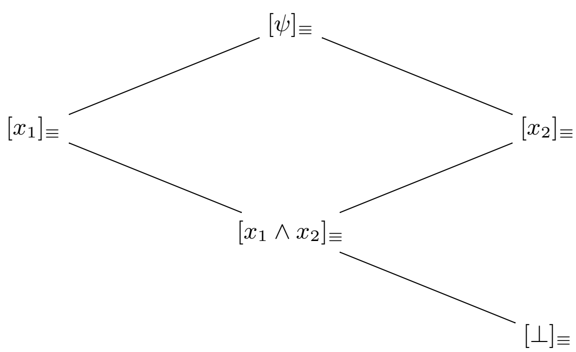


Fig. 3: A quotient of the free 2-generated Gödel algebra.

$\leq$	$[x_1]_\equiv$	$[x_2]_\equiv$
$[x_1 \wedge x_2]_\equiv$	$\times$	$\times$
$[x_1]_\equiv$	$\times$	
$[x_2]_\equiv$		$\times$

$\leq$	m_1	m_2
g_1	$\times$	$\times$
g_2	$\times$	
g_3		$\times$

Figure 4 shows the concept lattice associated with the Gödel algebra $\mathbf{A} = \mathcal{G}_2/(\psi = \top)$.

The characterization of free finitely generated Gödel algebras is a well-investigated topic that is beyond the scope of this paper. A functional representation is given in [21], while [1] is a state-of-the-art treatise on representations of many-valued logics. For our purposes it is sufficient to know that [2] contains a recursive description of $\mathcal{G}_n$, together with normal forms for Gödel logic, while in [14] the authors provide a combinatorial method to generate $\mathcal{G}_n$ and its quotients.

A general procedure to associate formal concepts with Gödel logic formulæ can be sketched out, based on the preceding examples. Let $\varphi_1, \ldots, \varphi_m, \psi$ be Gödel logic formulæ over $\{x_1, \ldots, x_n\}$, with $m \geq 0$, and $n \geq 1$. Generate $\mathcal{G}_n$

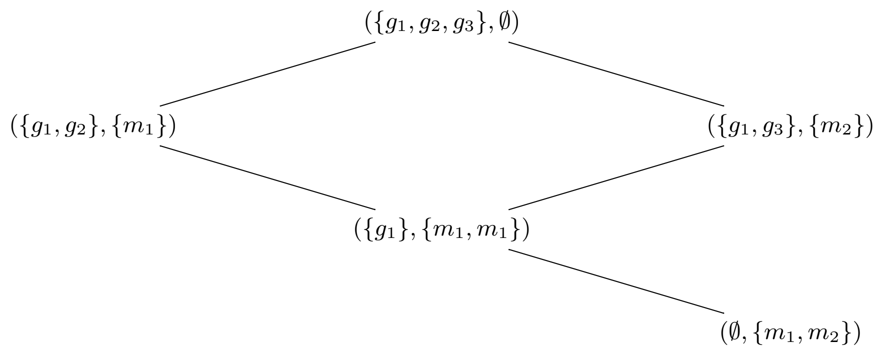


Fig. 4: The concept lattice associated with the Gödel algebra $\mathcal{G}_2/(\psi = \top)$

(see [2, 14]) and apply Proposition 1, obtaining $C_{\mathcal{G}_n}$. Then, $\{\varphi_1, \ldots, \varphi_m\} \vdash \psi$ amounts to evaluating ψ over $\mathcal{G}_n/(\varphi_1 = \top, \ldots, \varphi_m = \top)$. Proposition 2 states that $\mathbf{C}_{\mathcal{G}_n}$ is isomorphic to $\mathcal{G}_n$. Hence, such evaluation provides also a concept in $\mathbf{C}_{\mathcal{G}_n}$, that is, precisely the concept associated with ψ. This allows us to express formal concepts associated with ψ, for every theory $\{\varphi_1, \ldots, \varphi_m\}$ in Gödel logic.

6 Concluding Remarks

In the basic setting of FCA (see Section 2) it is assumed that concepts are crisp. In the literature one can find several studies whose aim is the "fuzzification" of I, the relation between G and M. The first one being [10], while [8] and [7] are good overview of these investigations. A further generalization of this type of approach is given in [6], where the author considers both relation and order in FCA as defined over fuzzy sets (or residuated lattices in general). Our method diverges from those approaches. We exploit the classical notions of FCA to obtain new insight on algebraic semantics of many-valued logics. Indeed, in the above sections we have shown that it is possible to associate a formal concept with every formula of Gödel logic. Further, we have provided a characterization of concept lattices isomorphic to Gödel algebras in terms of formal contexts. In this way we could effectively find contexts over which Gödel logic can be used to reason about.

In other words, whenever a concept lattice satisfies Proposition 3, we are dealing with a Gödel algebra of concepts. Under such conditions, concepts can be combined via the lattice operators meet and join – see (2) –, but also via the operations of p-implication and p-complement introduced in Section 3. The latter operations correspond, respectively, with the Gödel logic implication and negation, as shown in Proposition 2 and Corollary 1. In this sense we can say that our new interpretation can be viewed as an alternative semantics for Gödel logic. In order to acquire a full understanding of this semantics, we aim to investigate, in future work, the effect of the p-implication and p-complement over concepts obtained from contexts describing real-world scenarios. The ultimate goal is to get more insight about the meaning of Gödel logic by running empirical experiments over real data. Through this work we believe that this can be done.

The approach used in this work is not limited to Gödel logic, but it can be generally applied to many non-classical logics. Broadly speaking, it is sufficient that the corresponding algebraic semantics has a complete lattice reduct. As a many-valued logic, Gödel logic is a schematic extension of the fundamental system BL introduced by Hájek in [22], which in turn is a schematic extension of the *Monoidal T-norm Logic* (MTL) [16]. Hence, we believe that extending our method to other logics in this hierarchy could be an interesting task. The first issue to deal with is the fact that these logics have a monoidal conjunction in addition to the lattice one. A good starting point would be investigate logics where representations of free algebras are already available, *e.g.*, Nilpotent Minimum logic [11, 3], or Revised Drastic Product logic [27]. Further, many-valued logics are just particular substructural logics whose algebraic semantics is provided by the class of residuated lattices [19], giving thus space for further generalizations.

Additional research has to be done to compare our method with other investigations regarding alternative semantics and intended meaning of many-valued logics. For the former we can cite probabilistic [4, 3], temporal [9] and game-theoretic [17] approaches, and [24, 25, 13, 12] for the latter.

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