

Chapter 4

Partitions of a Finite Partially Ordered Set

Contributed Chapter

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Abstract In this paper, we investigate the notion of partition of a finite partially ordered set (poset, for short). We will define three different notions of partition of a poset, namely, monotone, regular, and open partition. For each of these notions we will find three equivalent definitions, that will be shown to be equivalent. We start by defining partitions of a poset in terms of fibres of some surjection having the poset as domain. We then obtain combinatorial characterisations of such notions in terms of blocks, without reference to surjection. Finally, we give a further, equivalent definition of each kind of partition by means of analogues of equivalence relations.

4.1 Introduction

A partition of a set A is a collection of nonempty, pairwise disjoint subsets, often called *blocks*, whose union is A . Equivalently, partitions can be defined by means of *equivalence relations*, by saying that a partition of a set A is the set of equivalence classes of an equivalence relation on A . A third definition of a partition can be given in terms of *fibres* of a surjection: a partition of a set A is the set $\{f^{-1}(y) \mid y \in B\}$ of fibres of a surjection $f : A \rightarrow B$.

In this paper, we investigate the notion of partition of a finite poset. Our goal is to find the analogues of the three definitions of partition of a set mentioned above, in terms of blocks, relations, and fibres, respectively. For the rest of this paper, we shall omit the term finite, as we only deal with finite posets; ‘poset partition’ means ‘partition of a poset’.

We begin our study of poset partitions with the notion of partition given in terms of fibres. Some background in category theory, and a few preliminary results on two

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different categories having posets as objects (Section 4.2), will allow us to identify three kind of morphisms appropriate to introduce our first definition. In Section 4.3, we introduce monotone, regular, and open partitions, according to the morphisms we are considering.

The definitions given in Section 4.3 need to mention morphism to describe what a poset partition is. To investigate the combinatorial structure of a poset partition, it is useful to have intrinsic notions of poset partitions making no reference to morphisms. In Section 4.4 we obtain such a combinatorial characterisation of each kind of partition.

In Section 4.5, we investigate the analogues of the definition of partition of a set given by means of equivalence relations. Again, we obtain characterisations of monotone, regular, and open partitions in this terms.

This piece of work contains a revisited and extended version of some of the results obtained in [3], where the author defines and analyses the notions of monotone and regular partitions of a poset, and in [4], where open partitions are introduced.

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4.2 Background, and Preliminary Results

When one defines a partition of a set in terms of fibres, one makes use of a special class of morphisms of the category¹ \mathbf{Set} of sets and functions. In fact, such definition exploits the notion of surjection, which can be shown to coincide in \mathbf{Set} with the notion of *epimorphism*. Recall that an epimorphism in a category is a morphism $f : A \rightarrow B$ that is right-cancellable with respect to composition: whenever $h \circ f = k \circ f$, for $h, k : B \rightarrow C$, we have $h = k$. The category-theoretic dual of the notion of epimorphism is *monomorphism*. In \mathbf{Set} , monomorphisms coincide with injections. The well-known fact that each function between sets factorises (in an essentially unique way) as a surjection followed by an injection can be reformulated in categorical terms by saying that the classes of epimorphisms and monomorphism form a factorisation system for \mathbf{Set} , or, equivalently, that $(\mathbf{epi}, \mathbf{mono})$ is a factorisation system for \mathbf{Set} . Epimorphism and factorisation systems will play a key role in the following.

First, consider the category \mathbf{Pos} of posets and *order-preserving maps* (also called *monotone maps*), *i.e.*, functions $f : P \rightarrow Q$, with P, Q posets such that $x \leq y$ in P implies $f(x) \leq f(y)$ in Q , for each $x, y \in P$. In \mathbf{Pos} , $(\mathbf{epi}, \mathbf{mono})$ is not a factorisation system; to obtain one we need to isolate a subclass of epimorphisms, called

¹ For background on category theory we refer, *e.g.*, to [1].

regular epimorphisms. A morphism $e : B \rightarrow C$ in an arbitrary category is a *regular epimorphism* if and only if there exists a pair $f, g : A \rightarrow B$ of morphisms such that

- (1) $e \circ f = e \circ g$,
- (2) for any morphism $e' : B \rightarrow C'$ with $e' \circ f = e' \circ g$, there exists a unique morphism $\psi : C \rightarrow C'$ such that $e' = \psi \circ e$.

Regular epimorphisms are epimorphisms, as one easily checks. While in **Set** regular epimorphisms and epimorphisms coincide, that is not the case in **Pos**. The dual notion of regular epimorphism (obtained by reversing arrows in the above statement) is *regular monomorphism*. It can be shown (see, e.g., [3, Proposition 2.5]) that **(regular epi, mono)** is a factorisation system for the category **Pos**. A second factorisation system for **Pos** is given by the classes of epimorphisms and regular monomorphisms. In other words, each order-preserving map between posets factorises in an essentially unique way both as a regular epimorphism followed by a monomorphism, and as an epimorphism followed by a regular monomorphism.

The existence of two distinct factorisation systems in **Pos** leads us to introduce two different notions of partition of a poset, one based on the use of epimorphisms, the other based on the use of regular epimorphisms.

Our next step is to characterise regular epimorphisms.

Notation. If π is a partition of a set A , and $a \in A$, we denote by $[a]_\pi$ the block of a in π . When no confusion is possible, we shall write $[a]$ instead of $[a]_\pi$. Further, let us stress our usage of different symbols for representing different types of binary relations. The symbol \leq denotes the partial order relation between elements of a poset. A second symbol, \triangleleft , represents the associated covering relation. Finally, the symbol \lesssim denotes *quasiorder relations*, sometimes called *preorders*, i.e. reflexive and transitive relations.

Definition 4.1 (Blockwise quasiorder). Let (P, \leq) be a poset and let π be a partition of the set P . For $x, y \in P$, x is *blockwise under y with respect to π* , written $x \lesssim_\pi y$, if and only if there exists a sequence $x = x_0, y_0, x_1, y_1, \dots, x_n, y_n = y \in P$ satisfying the following conditions.

- (1) For all $i \in \{0, \dots, n\}$, $[x_i] = [y_i]$.
- (2) For all $i \in \{0, \dots, n-1\}$, $y_i \leq x_{i+1}$.

Observe that the relation \lesssim_π in Definition 4.1 indeed is a quasiorder. In fact, if $x \leq y$ and $y \leq z$ for $x, y, z \in P$, then there exist two sequences $x = x_0, y_0, x_1, y_1, \dots, x_n, y_n = y$ and $y = y_{n+1}, z_{n+1}, y_{n+2}, z_{n+2}, \dots, y_{n+m}, z_{n+m} = z$ satisfying (1) and (2), and a sequence $x = x_0, y_0, x_1, y_1, \dots, x_n, y_n = y_{n+1}, z_{n+1}, y_{n+2}, z_{n+2}, \dots, y_{n+m}, z_{n+m} = z$ satisfying (1) and (2), too. Thus, $x \lesssim_\pi z$ and the relation \lesssim_π is transitive. The reflexivity of \lesssim_π results trivially.

The definition of blockwise quasiorder allows us to isolate a special kind of order-preserving map.

Definition 4.2 (Fibre-coherent map). Consider two posets P and Q . Let $f : P \rightarrow Q$ be a function, and let $\pi_f = \{f^{-1}(q) \mid q \in f(P)\}$ be the set of fibres of f . We say f is a *fibre-coherent map* whenever for any $p_1, p_2 \in P$, $f(p_1) \leq f(p_2)$ if and only if $p_1 \lesssim_{\pi_f} p_2$.

Proposition 4.1. In Pos , regular epimorphisms are precisely fibre-coherent surjections.

Proof. (\Rightarrow) Let (P, \leq_P) and (Q, \leq) be posets, let $e : P \rightarrow Q$ be a regular epimorphism and let $\pi_e = \{e^{-1}(q) \mid q \in Q\}$. Since e is epi, it is an order-preserving surjection. Moreover, by the definition of regular epimorphism, there exists a pair $f, g : R \rightarrow P$ of morphisms such that $e \circ f = e \circ g$.

Suppose, by way of contradiction, that e is not fibre-coherent. If $x \lesssim_{\pi_e} y$ for some $x, y \in P$, then there exists a sequence $x = x_0, y_0, x_1, y_1, \dots, x_n, y_n = y \in P$ satisfying conditions (1) and (2) in Definition 4.1. For such a sequence, since e is order-preserving, we have $e(x) = e(x_0) = e(y_0) \leq e(x_1) = e(y_1) \leq \dots \leq e(x_n) = e(y_n) = e(y)$. Thus, to satisfy the *absurdum hypothesis* there must exist $p_1, p_2 \in P$, with $e(p_1) = q_1$ and $e(p_2) = q_2$, such that $q_1 \leq q_2$ but $p_1 \not\lesssim_{\pi_e} p_2$. Note that p_1 and p_2 must be incomparable, and that $q_1 \neq q_2$.

Case (i). Suppose $q_1 \triangleleft q_2$, where \triangleleft is the covering relation induced by \leq . Consider the poset Q' having Q as underlying set, endowed with the relation \leq' obtained by removing from \leq the pair (q_1, q_2) . In other words, the only difference between \leq' and \leq is that $q_1 \leq q_2$, but $q_1 \not\leq' q_2$. Since $q_1 \triangleleft q_2$, removing (q_1, q_2) from \leq does not impair transitivity and \leq' indeed is a partial order.

Now, consider the function $e' : P \rightarrow Q'$ that coincides with e on the underlying sets. We want to show that e' is order-preserving. For this, let $x, y \in P$. It suffices to consider two cases only: $e(x) = q_1$ and $e(y) = q_2$, and viceversa. In any other case, e' preserves order just because e does. Suppose, without loss of generality, $x \in e^{-1}(q_1)$ and $y \in e^{-1}(q_2)$. Then, $x \not\leq_P y$, for else the chain p_1, x, y, p_2 would satisfy conditions (1) and (2) in Definition 4.1, contradicting $p_1 \not\lesssim_{\pi_e} p_2$. Moreover, $y \not\leq_P x$, because e is order preserving. Thus, for each $x \in e^{-1}(q_1)$ and $y \in e^{-1}(q_2)$, x and y are incomparable. Summing up, e' is order preserving. Since e' coincides with e on the underlying sets, we obtain $e' \circ f = e' \circ g$.

Since e is a regular epimorphism, by definition, we can find a unique morphism $\psi : Q \rightarrow Q'$ such that the diagram in Figure 4.1 commutes. Take $x, y \in P$ such

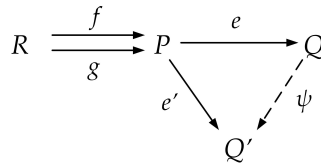


Fig. 4.1 Proof of Proposition 4.1.

that $e(x) = e'(x) = q_1$ and $e(y) = e'(y) = q_2$. Thus, we should have $\psi(q_1) = q_1$ and $\psi(q_2) = q_2$ but, by hypothesis, $q_1 \leq q_2$ and $q_1 \not\leq' q_2$, and such a ψ would not be order-preserving. Since f and g satisfy $e \circ f = e \circ g$ but are otherwise arbitrary, this would contradict the fact that e is a regular epimorphism. Therefore, e has to be fibre-coherent.

Case (ii). Suppose $q_1 \not\leq q_2$. Then there exists a sequence $k_1, k_2, \dots, k_u \in Q$ such that $q_1 = k_1 \triangleleft k_2 \triangleleft \dots \triangleleft k_u = q_2$. Let $x_1, x_2, \dots, x_u \in Q$ be such that $x_i \in e^{-1}(k_i)$, for each $i \in \{1, u\}$, and suppose $x_1 \lesssim_{\pi_e} x_2 \lesssim_{\pi_e} \dots \lesssim_{\pi_e} x_u$. Since $p_1 \in e^{-1}(k_1)$ and $p_2 \in e^{-1}(k_u)$ imply $p_1 \lesssim_{\pi_e} x_1$ and $x_u \lesssim_{\pi_e} p_2$, then, by transitivity, $p_1 \lesssim_{\pi_e} p_2$, contradicting our hypothesis. Thus, there exists an index j such that $k_j \triangleleft k_{j+1}$, but $x_j \not\lesssim_{\pi_e} x_{j+1}$. The proof follows now the same steps of Case (i), with x_j and x_{j+1} playing the role of p_1 and p_2 , respectively.

(\Leftarrow) Let (P, \leq_P) and (Q, \leq) be posets, and let $e : P \rightarrow Q$ be a fibre-coherent surjection. Consider the poset $R \subseteq P \times P$, having underlying set $\{(r_1, r_2) \in P \times P \mid e(r_1) = e(r_2)\}$, endowed with the order \leq_R defined by $(r_1, r_2) \leq_R (s_1, s_2)$ if and only if $r_1 \leq_P s_1$ and $r_2 \leq_P s_2$. Let $f, g : R \rightarrow P$ be the projection functions of R , i.e., f and g are the order-preserving maps such that, for each $r = (r_1, r_2) \in R$, $f(r) = r_1$, $g(r) = r_2$. Clearly, $e \circ f = e \circ g$.

We need to show that e is a regular epimorphism. Consider a poset (Q', \leq') and an order-preserving map $e' : P \rightarrow Q'$ such that $e' \circ f = e' \circ g$. Note that, for each $q \in Q$, if $x, y \in e^{-1}(q)$, there exists $r \in R$ such that $f(r) = x$ and $g(r) = y$. Thus, from $e' \circ f = e' \circ g$ follows that $e'(x) = e'(y)$. Since e is a surjection, we can construct a map $\psi : Q \rightarrow Q'$ by setting $\psi(q) = e'(x)$ for some $x \in e^{-1}(q)$, where $q \in Q$.

Let now $q_1, q_2 \in Q$ with $q_1 \leq q_2$, and let $x_1, x_2 \in P$ be such that $e(x_1) = q_1$, $e(x_2) = q_2$. By Definition 4.2, we have $x_1 \lesssim_{\pi_e} x_2$. Thus, by Definition 4.1 there exists a sequence $y_0, z_0, y_1, z_1, \dots, y_n, z_n \in P$ with $x_1 = y_0$ and $x_2 = z_n$ such that $e(y_i) = e(z_i)$, for $i = 0, \dots, n$, and $z_j \leq_P y_{j+1}$, for $j = 0, 1, \dots, n-1$. Moreover, from $e' \circ f = e' \circ g$ it follows that $e'(y_i) = e'(z_i)$, and, since e' is order-preserving, we have $e'(z_j) \leq' e'(y_{j+1})$. Thus, $e'(y_0) = e'(z_0) \leq' e'(y_1) = e'(z_1) \leq' \dots \leq' e'(y_n) = e'(z_n)$. Therefore, we have $\psi(q_1) = e'(x_1) \leq \psi(q_2) = e'(x_2)$ and ψ is order-preserving. The morphism ψ is now well defined and, by construction, satisfies $e'(x) = \psi(e(x))$ for all $x \in P$.

Let ψ' be another map from Q to Q' , $\psi \neq \psi'$, and let \bar{q} be an element of Q such that $\psi(\bar{q}) \neq \psi'(\bar{q})$. Since e is surjective, there exists $x \in P$ such that $e(x) = \bar{q}$. Then from $\psi'(\bar{q}) \neq \psi(\bar{q})$ and $e'(x) = \psi(\bar{q})$ we have $\psi'(\bar{q}) \neq e'(x)$ and $\psi' \circ e \neq e'$. Hence, $\psi : Q \rightarrow Q'$ is the unique function such that $\psi \circ e = e'$. Summing up, for an arbitrary morphism $e' : P \rightarrow Q'$ such that $e' \circ f = e' \circ g$, there exists a unique order preserving map $\psi : Q \rightarrow Q'$ such that $\psi \circ e = e'$, i.e., e is a regular epimorphism.

The second category we are going to consider is the category OPos of posets and open maps. Such maps arise naturally in the investigation of intuitionistic logic; cf. the notion of p-morphisms of Kripke frames, e.g. in [2]. An order-preserving function $f : P \rightarrow Q$ between posets is called *open* if whenever $f(u) \geq v'$ for $u \in P$ and $v' \in Q$, there is $v \in P$ such that $u \geq v$ and $f(v) = v'$. One can check that epimorphisms

in OPos are surjective open maps, and monomorphisms are injective open maps. As in Set, in OPos each epimorphism is a regular epimorphism. Further, OPos admits an (epi,mono) factorisation system. In the next section, we will see that surjective open maps in OPos induce a third kind of partition of a poset.

4.3 Partitions as Sets of Fibres

Poset partitions can be defined in terms of fibres. From the notions of epimorphism and regular epimorphism in Pos, we derive immediately the two following definitions.

Definition 4.3 (Monotone partition). A *monotone partition* of a poset P is a poset (π_f, \leq) , where π_f is the set of fibres² of an order-preserving surjection $f : P \rightarrow Q$, for some poset Q , and \leq is the partial order on π_f defined by

$$f^{-1}(q_1) \leq f^{-1}(q_2) \text{ if and only if } q_1 \leq q_2, \quad (4.1)$$

for each $q_1, q_2 \in Q$.

Definition 4.4 (Regular partition). A *regular partition* of a poset P is a poset (π_f, \leq) , where π_f is the set of fibres of a fibre-coherent surjection $f : P \rightarrow Q$, for some poset Q , and \leq is the partial order on π_f defined by

$$f^{-1}(q_1) \leq f^{-1}(q_2) \text{ if and only if } q_1 \leq q_2, \quad (4.2)$$

for each $q_1, q_2 \in Q$.

Remark 4.1. Since a fibre-coherent map is order-preserving, it follows immediately that each regular partition of a poset is a monotone partition.

Consider now the category OPos.

Definition 4.5 (Open partition). An *open partition* of a poset P is a poset (π_f, \leq) , where π_f is the set of fibres of a surjective open map $f : P \rightarrow Q$, for some poset Q , and \leq is the partial order on π_f defined by

$$f^{-1}(q_1) \leq f^{-1}(q_2) \text{ if and only if } q_1 \leq q_2, \quad (4.3)$$

for each $q_1, q_2 \in Q$.

Remark 4.2. One can check that an open map is fibre-coherent. It follows that each open partition of a poset is a regular partition, whence a monotone one.

There are regular partitions that are not open, and monotone partitions that are not regular; cf. Example 4.1.

² Note that, since f is surjective, π_f is a partition of the underlying set of P .

4.4 Partitions as Partially Ordered Sets of Blocks

For each definition in the previous section, we give a new definition in terms of partially ordered blocks without mentioning morphisms. We prove, for each notion of partition, that the two kinds of definition describe exactly the same objects.

Definition 4.6 (Monotone partition). A *monotone partition* of a poset P is a poset (π, \leq) where

- (i) π is a partition of the underlying set of P ,
- (ii) for each $p_1, p_2 \in P$, $p_1 \leq p_2$ implies $[p_1] \leq [p_2]$.

Theorem 4.1. *Definitions 4.3 and 4.6 are equivalent.*

Proof. (Definition 4.6 \Rightarrow Definition 4.3). Let π be a partition of the underlying set of a poset P , and let \leq be a partial order on π satisfying (ii). Consider the projection map $f : P \rightarrow \pi$ which sends each element of P to its block in π . Since a partition does not have empty blocks, f is a surjection. By (ii), f is order-preserving. Since, by construction, $f^{-1}(B) = B$ for each $B \in \pi$, the partial order \leq satisfies (4.1). Thus, (π, \leq) is a monotone partition of P according to Definition 4.3.

(Definition 4.3 \Rightarrow Definition 4.6). Let $f : P \rightarrow Q$ be an order-preserving surjection, and let (π_f, \leq) be a monotone partition of P , according to Definition 4.3. Since f is surjective, π_f is a partition of the underlying set of P . Consider $p_1, p_2 \in P$ such that $p_1 \leq p_2$. Since f is order-preserving, $f(p_1) \leq f(p_2)$ holds and, by (4.1), $[p_1] = f^{-1}(f(p_1)) \leq [p_2] = f^{-1}(f(p_2))$. We have so proved (ii). Thus, (π_f, \leq) is a monotone partition of P according to Definition 4.6.

Corollary 4.1. *Let (π, \leq) be a monotone partition of a poset P . Then, for each $p_1, p_2 \in P$,*

$$[p_1] = [p_2] \text{ if and only if } (p_1 \lesssim_{\pi} p_2 \text{ and } p_2 \lesssim_{\pi} p_1). \quad (4.4)$$

Proof. (\Rightarrow) Directly from Definition 4.1.

(\Leftarrow) Let $p_1, p_2 \in P$ be such that $p_1 \lesssim_{\pi} p_2$ and $p_2 \lesssim_{\pi} p_1$. Then, there exist two sequences $p_1 = x_0, y_0, x_1, y_1, \dots, x_n, y_n = p_2$ and $p_2 = z_0, w_0, \dots, z_m, w_m = p_1$ of elements of P satisfying Conditions (1) and (2) in Definition 4.1, with respect to π . By Condition (ii) in Definition 4.6, we have $[p_1] = [x_0] = [y_0] \leq [x_1] = [y_1] \leq \dots \leq [x_n] = [y_n] = [p_2]$, and $[p_2] = [z_0] = [w_0] \leq [z_1] = [w_1] \leq \dots \leq [z_m] = [w_m] = [p_1]$. Thus, $[p_1] = [p_2]$.

Definition 4.7 (Regular partition). A *regular partition* of a poset P is a poset (π, \leq) where

- (i) π is a partition of the underlying set of P ,
- (ii) for each $p_1, p_2 \in P$, $p_1 \lesssim_{\pi} p_2$ if and only if $[p_1] \leq [p_2]$.

Theorem 4.2. *Definitions 4.4 and 4.7 are equivalent.*

Proof. (Definition 4.7 \Rightarrow Definition 4.4). Let π be a partition of the underlying set of a poset P and let \leq be a partial order on π satisfying (ii). Consider the projection map $f : P \rightarrow \pi$ which sends each element of P to its block in π . Clearly, f is a surjection. Since, for all $p \in P$, $f(p) = [p]$, Condition (ii) is equivalent to the fibre-coherent condition in Definition 4.2. Thus, f is a fibre-coherent surjection, having π as its set of fibres. Moreover, since $f^{-1}(B) = B$ for all $B \in \pi$, the partial order \leq satisfies (4.2). Thus, (π, \leq) is a regular partition of P according to Definition 4.4.

(Definition 4.4 \Rightarrow Definition 4.7). Let $f : P \rightarrow Q$ be a fibre-coherent surjection, and let (π_f, \leq) be a regular partition of P , according to Definition 4.4. Since f is surjective, π_f is a partition of the underlying set of P . Consider $p_1, p_2 \in P$. By Definition 4.2, $p_1 \lesssim_{\pi_f} p_2$ if and only if $f(p_1) \leq f(p_2)$. By Definition 4.4, $f(p_1) \leq f(p_2)$ if and only if $[p_1] = f^{-1}(f(p_1)) \leq [p_2] = f^{-1}(f(p_2))$. Thus, $p_1 \lesssim_{\pi_f} p_2$ if and only if $[p_1] \leq [p_2]$, and (ii) is proved. Thus, (π_f, \leq) is a regular partition of P according to Definition 4.7.

Corollary 4.2. *Let (π, \leq) be a partition of the underlying set of a poset P satisfying (4.4). Then, there exists exactly one regular partition of P having π as underlying set.*

Proof. Let π be a partition of the underlying set of a poset P satisfying (4.4). Define the binary relation \leq on π by prescribing that, for all $p_1, p_2 \in P$, $p_1 \lesssim_{\pi} p_2$ if and only if $[p_1] \leq [p_2]$. It is straightforward to check that \leq is a partial order on P . By Definition 4.7, (π, \leq) is a regular partition of P . To see that it is the unique regular partition of P having π as underlying set, just observe that \leq must be a partial order satisfying Condition (ii) in Definition 4.7.

The uniqueness property of regular partitions proved in the above corollary does not hold, in general, for monotone partitions; cf. Figure 4.2, which shows three distinct monotone partitions of a given poset P having the same underlying set.

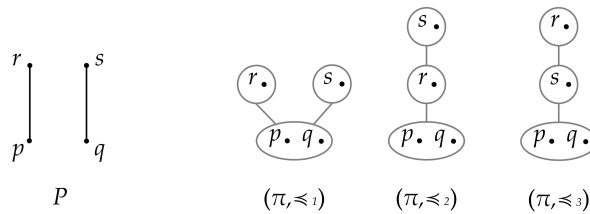


Fig. 4.2 Distinct monotone partitions with the same support π .

Given a poset P and a subset $S \subseteq P$, the *lower set* generated by S is

$$\downarrow S = \{p \in P \mid p \leq s, \text{ for some } s \in S\}.$$

Analogously, the *upper set* generated by S is $\uparrow S = \{p \in P \mid p \geq s, \text{ for some } s \in S\}$. When S is a singleton $\{s\}$, we write $\downarrow s$ for $\downarrow \{s\}$, and $\uparrow s$ for $\uparrow \{s\}$.

Definition 4.8 (Open partition). An *open partition* of a poset P is a poset (π, \leq) where

- (i) π is a partition of the underlying set of P such that for each $B \in \pi$ there exist $B_1, \dots, B_k \in \pi$ satisfying ³

$$\uparrow B = B_1 \cup \dots \cup B_k. \quad (4.5)$$

- (ii) for each $B, C \in \pi$, $B \leq C$ if and only if there are $p_1 \in B$, $p_2 \in C$ such that $p_1 \leq p_2$.

Theorem 4.3. *Definitions 4.5 and 4.8 are equivalent.*

Proof. (Claim). If π satisfies (i), then, for each $B, C \in \pi$, $C \subseteq \uparrow B$ if and only if there are $p_1 \in B$, $p_2 \in C$ such that $p_1 \leq p_2$.

Whenever $p_1 \leq p_2$, the block C intersects the upper set of the block B . By (4.5), C must be entirely contained in $\uparrow B$. The converse is trivial.

(Definition 4.8 \Rightarrow Definition 4.5). Let π be a partition of the underlying set of a poset P satisfying (i) and let \leq be a partial order on π satisfying (ii).

Let us consider the projection map $f : P \rightarrow \pi$ which sends each element of P to its block. Let $p_1, p_2 \in P$. If $p_1 \leq p_2$ then, by (ii), $f(p_1) = [p_1] \leq f(p_2) = [p_2]$ and f is order-preserving. Since π does not have empty blocks, f is surjective. To show f is open, we consider $p_1 \in P$, and $B \in \pi$, such that $B \leq [p_1]$. By (ii) and (Claim), $[p_1] \subseteq \uparrow B$. Thus, there exists $p_2 \in B$ such that $p_2 \leq p_1$. Since $f(p_2) = [p_2] = B$, f is open. Therefore, (π, \leq) is a regular partition of P according to Definition 4.5.

(Definition 4.5 \Rightarrow Definition 4.8). Let $f : P \rightarrow Q$ be a surjective open map, and let (π_f, \leq) be an open partition of P , according to Definition 4.5. Suppose, by way of contradiction, that (4.5) does not hold. Thus, there exist $p_1, p_2 \in C$ such that $p_1 \in \uparrow B$, but $p_2 \notin \uparrow B$, for some $B, C \in \pi_f$. Let $f(B) = q$. Since f is order-preserving, $q \in \downarrow f(p_1)$. Since f is open, $q \notin \downarrow f(p_2)$, for else we would find $p \in B$ with $p \leq p_2$, against the hypothesis $p_2 \notin \uparrow B$. Since $f(p_1) = f(p_2)$, $q \in \downarrow f(p_1)$ and $q \notin \downarrow f(p_2)$ is a contradiction. Thus, (i) holds.

To prove (ii) consider $p_1, p_2 \in P$, and suppose $p_1 \leq p_2$. Since f is order-preserving, $f(p_1) \leq f(p_2)$. By Condition (4.3) in Definition 4.5, $[p_1] = f^{-1}(f(p_1)) \leq [p_2] = f^{-1}(f(p_2))$, and one side of (ii) is proved.

Suppose now that for some $B, C \in \pi_f$, $B \leq C$. Let $q_1 = f(B)$ and $q_2 = f(C)$. By Condition (4.3), $q_1 \leq q_2$ in Q . Since f is surjective, fibres are not empty, and there exists $p_1 \in B$. Moreover, since f is open, there exists $p_2 \in f^{-1}(q_1) = C$ such that $q_2 \leq q_1$. We have so proved (ii), and the proof is complete. Thus, (π_f, \leq) is an open partition of P according to Definition 4.8.

³ Here and in the following lower and upper sets are always relative to the order \leq of the poset.

Next we prove that each open partition is solely determined by its underlying set.

Corollary 4.3. *Let P be a poset, and let π be a partition of the underlying set of P satisfying (i) in Definition 4.8. Then, there exists a unique partial order \leq on P such that (π, \leq) is an open partition of π .*

Proof. (Claim). For each $B, C \in \pi$, $C \subseteq \uparrow B$ if and only if there are $p_1 \in B$, $p_2 \in C$ such that $p_1 \leq p_2$.

Whenever $p_1 \leq p_2$, the block C intersects the upper set of the block B . By (4.5), C must be entirely contained in $\uparrow B$. The converse is trivial.

Let \leq be the binary relation on π satisfying (ii) in Definition 4.8. Clearly, \leq is uniquely determined. We need to show that \leq is a partial order on π . Reflexivity and transitivity of \leq hold trivially. To show antisymmetry, let $B, C \in \pi$ be such that $B \leq C$ and $C \leq B$. Let $b_1 \in B$. Since $C \leq B$, by (Claim), there exists $c_1 \in C$ such that $c_1 \leq b_1$. Since $B \leq C$ there exists $b_2 \in B$ such that $b_2 \leq c_1 \leq b_1$. We can construct in this way an infinite chain $\dots \leq c_i \leq b_i \leq \dots \leq b_2 \leq c_1 \leq b_1$. Since P is finite, we can find an element $b_s \in B$ which occurs in the chain more than once, and, by construction, an element $c_t \in C$ such that $b_s \leq c_t \leq b_s$. Therefore, $b_s = c_t$. Since π is a set partition, and its blocks are pairwise disjoint, we obtain $B = C$. Thus, the relation \leq is antisymmetric; therefore it is a partial order on π .

We close this section with an example.

Example 4.1. We refer to Figure 4.3, and consider the poset P .

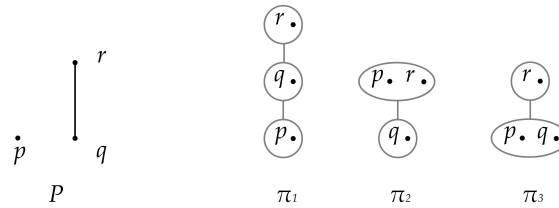


Fig. 4.3 Example 4.1.

One can easily check, using the characterisations of poset partitions provided in Definitions 4.6, 4.7, and 4.8, that the following hold.

- π_1 is a monotone partition of P , but it is not regular.
- π_2 is a regular partition of P , but it is not open.
- π_3 is an open partition of P .

4.5 Partitions Induced by Quasiorders

A quasiorder relation \lesssim on a set A induces on A an equivalence relation \equiv defined as

$$x \equiv y \text{ if and only if } x \lesssim y \text{ and } y \lesssim x, \text{ for any } x, y \in A. \quad (4.6)$$

The set π of equivalence classes of \equiv is a partition of A .

Notation. In the following we denote by $[x]_{\lesssim}$ the equivalence class (the block) of the element x induced by the quasiorder \lesssim via the equivalence relation defined in (4.6).

Further, the quasiorder \lesssim induces on π a partial order \leq defined by

$$x \leq y \text{ if and only if } [x]_{\lesssim} \leq [y]_{\lesssim}, \text{ for any } x, y \in A. \quad (4.7)$$

We call (π, \leq) *the poset of equivalence classes induced by \lesssim* .

This correspondence allows us to give a further definition of monotone, regular, and open partition of a poset by means of quasiorders.

Definition 4.9 (Monotone partition). A *monotone partition* of a poset (P, \leq) is the poset of equivalence classes induced by a quasiorder \lesssim on P extending \leq , *i.e.* such that $\leq \subseteq \lesssim$.

Theorem 4.4. *Definitions 4.6 and 4.9 are equivalent.*

Proof. (Definition 4.6 \Rightarrow Definition 4.9). Let π be a partition of the underlying set of a poset P , and let \leq be a partial order on π satisfying Condition (ii) in Definition 4.6. Consider the relation \lesssim on P defined by

$$p_1 \lesssim p_2 \text{ if and only if } [p_1]_{\pi} \leq [p_2]_{\pi}, \text{ for each } p_1, p_2 \in P. \quad (4.8)$$

One can easily check that \lesssim is reflexive and transitive, and thus it is a quasiorder on P . Moreover, $[p_1]_{\pi} = [p_2]_{\pi}$ if and only if $[p_1]_{\pi} \leq [p_2]_{\pi}$ and $[p_2]_{\pi} \leq [p_1]_{\pi}$. Thus, by (4.8), $[p_1]_{\pi} = [p_2]_{\pi}$ if and only if $p_1 \lesssim p_2$ and $p_2 \lesssim p_1$. By (4.6), $[p_1]_{\pi} = [p_2]_{\pi}$ if and only if $[p_1]_{\lesssim} = [p_2]_{\lesssim}$. Therefore, π coincide with the set of equivalence classes induced by \lesssim . Moreover, by (4.8) and (4.7), \leq coincide with the partial order on π induced by \lesssim . Thus, (π, \leq) is the poset of equivalence classes of P induced by \lesssim .

Suppose now that $p_1 \leq p_2$, for some $p_1, p_2 \in P$. By (ii) in Definition 4.6 we have $[p_1]_{\pi} \leq [p_2]_{\pi}$. By (4.8), $p_1 \lesssim p_2$. Thus, \lesssim extends \leq , and one side of the statement is proved.

(Definition 4.9 \Rightarrow Definition 4.6). Let (P, \leq) be a poset, let \lesssim be a quasiorder on P extending \leq , and let (π, \leq) be the poset of equivalence classes induced by \lesssim . Since \lesssim extends \leq , for each $p_1, p_2 \in P$, $p_1 \leq p_2$ implies $p_1 \lesssim p_2$. Moreover, by (4.7), $p_1 \lesssim p_2$ implies $[p_1]_{\lesssim} \leq [p_2]_{\lesssim}$. Thus, Condition (ii) in Definition 4.6 hold. We obtain that (π, \leq) is a monotone partition of P according to Definition 4.6.

For what regular partitions are concerned, the definition in terms of quasiorders can be formulated as follows.

Definition 4.10 (Regular partition). A *regular partition* of a poset (P, \leq) is the poset of equivalence classes induced by a quasiorder \lesssim on P extending \leq and satisfying

$$\lesssim = \text{tr}(\leq \setminus \rho), \quad (4.9)$$

where $\text{tr}(R)$ denotes the transitive closure of the relation R , and ρ is a binary relation defined by

$$\rho = \{(x, y) \in P \times P \mid x \lesssim y, x \not\lesssim y, y \not\lesssim x\}.$$

Theorem 4.5. *Definitions 4.7 and 4.10 are equivalent.*

Proof. (Definition 4.7 \Rightarrow Definition 4.10). Let π be a partition of the underlying set of a poset P , and let \leq be a partial order on π satisfying Condition (ii) in Definition 4.7. Consider the relation \lesssim on P defined by

$$p_1 \lesssim p_2 \text{ if and only if } [p_1]_\pi \leq [p_2]_\pi, \text{ for each } p_1, p_2 \in P. \quad (4.10)$$

Since (π, \leq) is a monotone partition of P , proceeding as in the first part of the proof of Theorem 4.4, we obtain that (π, \leq) is the poset of equivalence classes induced by \lesssim , and that \lesssim extends \leq . Moreover, by Condition (ii) in Definition 4.7, \lesssim coincides with \lesssim_π .

It remains to prove that \lesssim satisfies (4.9). Let $p_1, p_2 \in P$ be such that $p_1 \lesssim p_2$. If $(p_1, p_2) \notin \rho$, then $(p_1, p_2) \in \text{tr}(\leq \setminus \rho)$, trivially. Let then $(p_1, p_2) \in \rho$, that is $p_1 \lesssim p_2$, $p_2 \not\lesssim p_1$, $p_1 \not\lesssim p_2$. Since \lesssim coincides with \lesssim_π , there exists a sequence $p_1 = x_0, y_0, x_1, y_1, \dots, x_n, y_n = p_2$ satisfying Conditions (1) and (2) in Definition 4.1. By Condition (1) in Definition 4.1, and by (4.6), for each $i \in \{0, \dots, n\}$, $x_i \lesssim y_i$ and $y_i \lesssim x_i$ hold. Thus, $(x_i, y_i) \notin \rho$. By Condition (2) in Definition 4.1, and since \lesssim extends \leq , for each $i \in \{0, \dots, n-1\}$, we have that $y_i \lesssim x_{i+1}$. Thus, $(y_i, x_{i+1}) \notin \rho$. Summing up, no pair of adjacent element in the sequence $x_0, y_0, x_1, y_1, \dots, x_n, y_n$ belongs to ρ . Since each of these pairs belong to \lesssim , $(p_1, p_2) \in \text{tr}(\leq \setminus \rho)$. Thus, (4.9) holds, and the first part of the statement is settled.

(Definition 4.10 \Rightarrow Definition 4.7). Let (P, \leq) be a poset, let \lesssim be a quasiorder on P extending \leq and satisfying (4.9) and let (π, \leq) be the poset of equivalence classes induced by \lesssim .

(Claim 1). $p_1 \lesssim_\pi p_2$ implies $p_1 \lesssim p_2$, for each $p_1, p_2 \in P$.

Suppose that $p_1 \lesssim_\pi p_2$, for some $p_1, p_2 \in P$. Thus, there exists a sequence $p_1 = x_0, y_0, x_1, y_1, \dots, x_n, y_n = p_2 \in P$ satisfying Conditions (1) and (2) in Definition 4.1. By Condition (1), for each $i \in \{0, \dots, n\}$, $[x_i]_\pi = [y_i]_\pi$. Thus, by (4.7), $x_i \lesssim y_i$. By Condition (2), for each $i \in \{0, \dots, n-1\}$, $y_i \leq x_{i+1}$. Since, by hypothesis, \lesssim extends \leq , we have $y_i \lesssim x_{i+1}$. Therefore, whenever $p_1 \lesssim_\pi p_2$, we have $p_1 = x_0 \lesssim y_0 \lesssim \dots \lesssim x_n \lesssim y_n = p_2$. By transitivity $p_1 \lesssim p_2$.

(Claim 2). $p_1 \lesssim p_2$ implies $p_1 \lesssim_\pi p_2$, for each $p_1, p_2 \in P$.

Let $p_1 \lesssim p_2$, for some $p_1, p_2 \in P$. We shall analyze three different cases, covering all the possibilities for p_1 and p_2 .

Case (a). $p_2 \lesssim p_1$. Since $p_1 \lesssim p_2$ and $p_2 \lesssim p_1$, by (4.6), we have $[p_1]_\pi = [p_2]_\pi$. Thus, the sequence p_1, p_2 satisfies Conditions (1) and (2) in Definition 4.1, with respect to π . We obtain $p_1 \lesssim_\pi p_2$.

Case (b). $p_2 \not\lesssim p_1$, and $p_1 \lesssim p_2$. Since the sequence p_1, p_1, p_2, p_2 satisfies Conditions (1) and (2) in Definition 4.1, we have $p_1 \lesssim_\pi p_2$.

Case (c). $p_2 \not\lesssim p_1$, $p_1 \not\lesssim p_2$, that is, $(p_1, p_2) \in \rho$. Let $R = \lesssim \setminus \rho$. Since $p_1 \lesssim p_2$, the pair (p_1, p_2) arises in \lesssim from the transitive closure of the binary relation R . Thus, there exists a sequence $p_1 = z_0 \lesssim z_1 \lesssim \cdots \lesssim z_r = p_2$ of elements of P such that, for all $i \in \{0, \dots, r-1\}$, $z_i R z_{i+1}$. For each of this pair of elements, either Case (a) or Case (b) apply. Thus, $z_i \lesssim_\pi z_{i+1}$. Since \lesssim_π is transitive, we obtain immediately $p_1 \lesssim_\pi p_2$.

In any case, whenever $p_1 \lesssim p_2$, we have $p \lesssim_\pi q$, and the claim is settled.

By (Claim 1) and (Claim 2), the quasiorders \lesssim and \lesssim_π coincide, and Condition (ii) in Definition 4.7 is trivially verified. We obtain that (π, \lesssim) is a regular partition of P according to Definition 4.7.

Remark 4.3. Let (π, \lesssim) be the monotone partition induced by a quasiorder \lesssim . By the construction given in the proof of Theorem 4.5, we infer that (π, \lesssim) is a regular partition if and only if \lesssim coincides with \lesssim_π .

In the case of open partitions, the definition in terms of quasiorders is as follows.

Definition 4.11 (Open partition). An *open partition* of a poset (P, \leq) is the poset of equivalence classes induced by a quasiorder \lesssim on P extending \leq and such that, for every $p, q \in P$,

$$\text{if } p \lesssim q, \text{ then there exists } p' \in P \text{ such that } p \lesssim p', p' \lesssim p, \text{ and } p' \leq q. \quad (4.11)$$

Theorem 4.6. *Definitions 4.8 and 4.11 are equivalent.*

Proof. (Definition 4.8 \Rightarrow Definition 4.11). Let (π, \lesssim) be an open partition of a poset P , satisfying thus Conditions (i) and (ii) in Definition 4.8. Consider the relation \lesssim on P defined by

$$p_1 \lesssim p_2 \text{ if and only if } [p_1]_\pi \leq [p_2]_\pi, \text{ for each } p_1, p_2 \in P. \quad (4.12)$$

Since (π, \lesssim) is a monotone partition of P , proceeding as in the first part of the proof of Theorem 4.4, we obtain that (π, \lesssim) is the poset of equivalence classes induced by \lesssim , and that \lesssim extends \leq . Let $p \lesssim q$ for some $p, q \in P$. By (4.12), $[p]_\pi \leq [q]_\pi$. By Conditions (i) and (ii) in Definition 4.8, $[p]_\pi \leq [q]_\pi$ implies $[q]_\pi \subseteq \uparrow [p]_\pi$. Thus, there exists $p' \in [p]_\pi$ such that $p' \leq q$, and (4.11) is satisfied. Therefore, (π, \lesssim) is an open partition of P according to Definition 4.11.

(Definition 4.11 \Rightarrow Definition 4.8). Let (π, \lesssim) be the poset of equivalence classes induced by \lesssim .

(Claim). Let $B, C \in \pi$. Then, $B \lesssim C$ if and only if $C \subseteq \uparrow B$.

(\Leftarrow) Suppose $C \subseteq \uparrow B$. Since blocks are nonempty, there exist $p \in B$ and $q \in C$ such that $p \leq q$. Since \lesssim extends \leq , we have $p \lesssim q$. Hence, by (4.7), $B \lesssim C$.

(\Rightarrow) Suppose $B \lesssim C$. By (4.7), for each $q \in C$, $p \in B$, we have $p \lesssim q$. Moreover, by (4.11), there exists $p' \in B$ such that $p' \leq q$. Thus, for each $q \in C$ there exists $p' \in B$ such that $p' \leq q$. In other words, $C \subseteq \uparrow B$.

Conditions (ii) in Definition 4.8 follows immediately from (Claim). Let $B, C \in \pi$, and suppose that $p \leq q$, for some $p \in B$, $q \in C$. Since \lesssim extends \leq , we have $p \lesssim q$. Moreover, by (4.7), $B \lesssim C$. By (Claim), C is entirely contained in $\uparrow B$. We have so proved that whenever a block C intersects $\uparrow B$, the block C is entirely contained in $\uparrow B$. Thus, $\uparrow B$ can be written as a union of blocks of π , as in Conditions (i) in Definition 4.8, and the theorem is proved.

4.6 Further Work

The further development of a theory of partitions of finite posets can follow several research direction. One of these concerns enumeration problems. It is easy to realize that enumerating poset partitions is never a simple problem, except in trivial cases – *e.g.* counting monotone, regular, and open partitions of chains, or of trivially ordered posets. Indeed, as we have shown, the enumeration of poset partitions is tightly related to the problem of counting the number of quasiorders on a finite set of points. Special cases may be tractable; we plan to address the issue in future work, following the relatively simple cases investigated in [3].

Another direction the we plan to follow in our future research involves the investigation of the ordered structure of all poset partitions of a poset. In [3], we proved that the classes of all monotone and regular partitions of a poset form a lattice, and we analysed its first properties.

Applications of the notions of monotone, regular, and open partitions can also be found in the study of distributive lattices, and Heyting algebras. Indeed, we recall here the well-known fact that the category Pos is dually equivalent to the category of finite bounded distributive lattices with their $\{0, 1\}$ -preserving lattice homomorphisms (for details see, *e.g.*, [5]). In the dual category of finite distributive lattices our notions of monotone and regular partition correspond precisely to the notions of sublattice and regular sublattice (see [3, Chapter 5]), respectively, of a distributive lattice.

Concerning open partitions, the category OPos can be proved to be dually equivalent to the category of finite Heyting algebras with their homomorphisms (for details on Heyting algebras, see, *e.g.*, [6]). In the dual category of finite Heyting algebras our notion of open partition corresponds precisely to the notion of subalgebra.

The above duality allows, for instance, the application of our results on open partitions to the study of the notion of probability distribution in Gödel logic (an extension of the intuitionistic propositional calculus); please see [4].

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