Indiscernibility Relations on Partially Ordered Sets

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***Abstract*—Let the pair (U,A) be an information system, where U is a collection of objects, the universe, and A is a finite set of attributes. If we consider a subset B of the set of attributes A, we can associate with B an indiscernibility relation on U, and thus a partition of the set U.**

Endow U with a partial order, obtaining a partially ordered set P, and consider an information system having P as universe. In this piece of work we investigate the notion of indiscernibility relation on a such information system. In particular, we introduce the notion of compatibility between an indiscernibility relation I on U and the partially ordered set P, and we establish a criterion for I to be compatible with P.

***Keywords*-Rough Set; Indiscernibility Relation; Partially Ordered Set; Partition.**

I. Introduction

Rough sets were introduced in the early 1980s by Pawlak ([15]). Since then, lot of work has been published, developing and enriching the theory of rough sets (see, *e.g.*, [9], [13], [17], [18], [19], [21]), and showing how the notion of rough set is suited to solve several problems in different fields of application (see, *e.g.*, [4], [5], [6], [7], [8]).

The notion of indiscernibility relation (see, *e.g.*, [12], [16]) stand at the basis of the theory of rough sets, as stated by Pawlak in [16]:

> "The rough set philosophy is founded on the assumption that with every object of the universe of discourse we associate some information (data, knowledge) [...]. Objects characterized by the same information are indiscernible (similar) in view of the available information about them. *The indiscernibility relation generated in this way is the mathematical basis of rough set theory*."

The focus of this work is the study of the indiscernibility relation in an information system where the universe we are considering is partially ordered.

Some of the mathematical concept used in this paper has been developed by the author in [1] and [2]. In these works, amongst other results, starting from a perspective from category theory, the author provides two different notions of partition of a partially ordered set (*poset*, for short), namely, *monotone partition* and *regular partition*.

In this paper, in Section II, we present some results obtained in those works, giving the necessary background.

In Section III, we analyse the connection between monotone and regular partitions of a poset, and indiscernibility relations. To this end, we will consider an information system $\mathcal{P} = (P, \bar{A})$ having as universe a finite poset P. Then, we will introduce a notion of compatibility between the poset P, and the (set) partition π induced on the underlying set of P via an indiscernibility relation, and prove a criterion for an indiscernibility relation to be compatible with P. When the compatibility criterion is satisfied, an extension of π to a monotone (or regular) partition of P is possible.

Finally, in Section IV, we present a concrete example in which the universe of an information system can be endowed with a partial order in a natural way. Referring to this example, we will see how to apply the criterion presented in Section III to see which subsets of attributes generates indiscernibility relations compatible with the (partially ordered) universe under consideration.

II. Partitions of a Partially Ordered Set

A partition of a set A is a collection of nonempty, pairwise disjoint subsets, often called *blocks*, whose union is A. Equivalently, partitions can be defined by means of *equivalence relations*, by saying that a partition of a set A is the set of equivalence classes of an equivalence relation on A. A third definition of a partition can be given in terms of *fibres* of a surjection: a partition of a set A is the set $\{f^{-1}(y) \mid y \in B\}$ of fibres of a surjection $f : A \to B$. (For background on classical theory of partitions, see [11].)

In this section, we show how to obtain feasible notions of partition of a finite poset. We start by providing definitions of partition in terms of *fibres*. Such kind of definition arise naturally when thinking in terms of categories, *i.e.*, in terms of objects and maps between them. Then, we will provide the corresponding definitions in terms of *blocks*, and in terms of *relations*. Full proofs of the results presented in this section are contained in [1] and [2].

A. Background

Some categorical notions are needed to present our definitions of partition of a poset. For background on category theory we refer, *e.g.*, to [10]. Let Set be the category having sets as objects and functions as morphisms. Recall that an *epimorphism* in a category is a morphism $f : A \to B$ that is right-cancelable with respect to composition: whenever $h \circ f = k \circ f$, for $h, k : B \to C$, we have $h = k$. The category-theoretic dual of the notion of epimorphism is *monomorphism*. Denote by epi the class of all epimorphisms in a category, and by mono the class of all monomorphisms.

To define a partition of a set in terms of fibres, one makes use of a special class of morphisms of the category Set. In fact, such definition exploits the notion of surjection, which can be shown to coincide in Set with the notion of *epimorphism*. Moreover, in Set, monomorphisms coincide with injections. The well-known fact that each function factorises (in an essentially unique way) as a surjection followed by an injection can be reformulated in categorical terms by saying that the epi and mono form a factorisation system for Set, or, equivalently, that (epi,mono) is a factorisation system for Set. Epimorphisms and factorisation systems will play a key role in the following.

Consider the category Pos of posets and *order-preserving maps* (also called *monotone maps*), *i.e.*, functions $f : P \to Q$, with P, Q posets, such that $x \leqslant y$ in P implies $f(x) \leqslant f(y)$ in Q, for each $x, y \in P$. In Pos, (epi,mono) is not a factorisation system; to obtain one we need to isolate a subclass of epimorphisms, called regular epimorphisms. A morphism $e : B \to C$ in an arbitrary category is a *regular epimorphism* if and only if there exists a pair $f, g : A \to B$ of morphisms such that

(1) $e \circ f = e \circ g$,

(2) for any morphism $e' : B \to C'$ with $e' \circ f = e' \circ g$, there exists a unique morphism $\psi : C \to C'$ such that $e' = \psi \circ e$.

Regular epimorphisms are epimorphisms, as one easily checks. While in Set regular epimorphisms and epimorphisms coincide, that is not the case in Pos. The dual notion of regular epimorphism (obtained by reversing arrows in the above statement) is *regular monomorphism*. It can be shown (see, *e.g.*, [1, Proposition 2.5]) that (regular epi,mono) is a factorisation system for the category Pos. A second factorisation system for Pos is given by the classes of epimorphisms and regular monomorphisms. In other words, each order-preserving map between posets factorises in an essentially unique way both as a regular epimorphism followed by a monomorphism, and as an epimorphism followed by a regular monomorphism.

The existence of two distinct factorisation systems in Pos leads us to introduce two different notions of partition of a poset, one based on the use of epimorphisms, the other based on the use of regular epimorphisms.

Our next step is to characterise regular epimorphisms.

Notation. If π is a partition of a set A, and $a \in A$, we denote by $[a]_\pi$ the block of a in π. When no confusion is possible, we shall write $[a]$ instead of $[a]_\pi$. Further, let us stress our usage of different symbols for representing different types of binary relations. The symbol $\leqslant$ denotes the partial order relation between elements of a poset. A second symbol, $\lesssim$, denotes *preorder relations*, sometimes called *quasiorders*, *i.e*, reflexive and transitive relations.

Definition 2.1 (Blockwise preorder): Let $(P, \leqslant)$ be a poset and let π be a partition of the set P. For $x, y \in P$, *x is blockwise under y with respect to π*, written $x \lesssim_\pi y$, if and only if there exists a sequence $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y \in P$ satisfying the following conditions.

(1) For all $i \in \{0, \ldots, n\}$, $[x_i] = [y_i]$.

(2) For all $i \in \{0, \ldots, n-1\}$, $y_i \leqslant x_{i+1}$.

Observe that the relation $\lesssim_\pi$ in Definition 2.1 indeed is a preorder. In fact, if $x \leqslant y$ and $y \leqslant z$ for $x, y, z \in P$, then there exist two sequences $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y$ and $y = y_{n+1}, z_{n+1}, y_{n+2}, z_{n+2}, \ldots, y_{n+m}, z_{n+m} = z$ satisfying (1) and (2), and a sequence $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y_{n+1}, z_{n+1}, y_{n+2}, z_{n+2}, \ldots, y_{n+m}, z_{n+m} = z$ satisfying (1) and (2), too. Thus, $x \lesssim_\pi z$ and the relation $\lesssim_\pi$ is transitive. The reflexivity of $\lesssim_\pi$ results trivially.

The definition of blockwise preorder allows us to isolate a special kind of order-preserving map.

Definition 2.2 (Fibre-coherent map): Consider two posets P and Q. Let $f : P \to Q$ be a function, and let $\pi_f = \{f^{-1}(z) \mid z \in f(P)\}$ be the set of fibres of f. We say f is a *fibre-coherent map* whenever for any $x, y \in P$, $f(x) \leqslant f(y)$ if and only if $x \lesssim_{\pi_f} y$.

Proposition 2.3: In Pos, regular epimorphisms are precisely fibre-coherent surjections.

Proof: See [2, Proposition 4.1]. ■

B. Partitions as Sets of Fibres

Poset partitions can be defined in terms of fibres. From the notions of epimorphism and regular epimorphism in Pos, we derive immediately the two following definitions.

Definition 2.4 (Monotone partition): A *monotone partition* of a poset P is a poset $(\pi_f, \leqslant)$, where π_f is the set of fibres[1] of an order-preserving surjection $f : P \to Q$, for some poset Q, and $\leqslant$ is the partial order on π_f defined by

$$f^{-1}(x) \leqslant f^{-1}(y) \text{ if and only if } x \leqslant y\,, \tag{1}$$

for each $x, y \in Q$.

Definition 2.5 (Regular partition): A *regular partition* of a poset P is a poset $(\pi_f, \leqslant)$, where π_f is the set of fibres of

[1]Note that, since f is surjective, π_f is a partition of the underlying set of P.

a fibre-coherent surjection $f : P \to Q$, for some poset Q, and $\preccurlyeq$ is the partial order on π_f defined by

$$f^{-1}(x) \preccurlyeq f^{-1}(y) \text{ if and only if } x \leqslant y, \tag{2}$$

for each $x, y \in Q$.

Since a fibre-coherent map is order-preserving, it follows immediately that each regular partition of a poset is a monotone partition. Clearly, there are monotone partitions that are not regular; cf. Example 1.

Remark 1: Another, distinct, notion of partition of a poset can be derived by taking into account the category of posets and *open maps*, instead of the category Pos we are considering. Such kind of partition is called *open partition* (see [2, Definition 4.8]). An application of the notion of open partition can be found in [3].

C. Partitions as Partially Ordered Sets of Blocks

For each definition in the previous section, we give the corresponding definition in terms of partial orders on blocks, without mentioning morphisms.

Definition 2.6 (Monotone partition): A *monotone partition* of a poset P is a poset $(\pi, \preccurlyeq)$ where

(i) π is a partition of the underlying set of P,
(ii) for each $x, y \in P$, $x \leqslant y$ implies $[x] \preccurlyeq [y]$.

Definition 2.7 (Regular partition): A *regular partition* of a poset P is a poset $(\pi, \preccurlyeq)$ where

(i) π is a partition of the underlying set of P,
(ii) for each $x, y \in P$, $x \lesssim_\pi y$ if and only if $[x] \preccurlyeq [y]$.

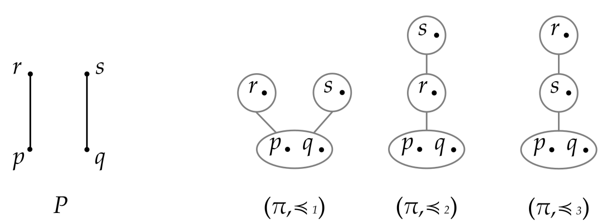

Figure 1. Distinct monotone partitions with the same support π.

Figure 1 shows three distinct monotone partitions of a given poset P having the same underlying set. We close this subsection with an example.

Example 1: We refer to Figure 2, and consider the poset P. One can easily check, using the characterisations of poset partitions provided in Definitions 2.6 and 2.7, that the following hold.

- π_1 is a monotone partition of P, but it is not regular.
- π_2 and π_3 are regular partitions of P, thus monotone ones.

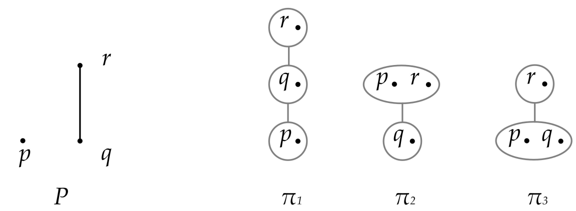

Figure 2. Example 1.

D. Partitions Induced by Preorders

A preorder relation $\lesssim$ on a set A induces on A an equivalence relation $\equiv$ defined as

$$x \equiv y \text{ if and only if } x \lesssim y \text{ and } y \lesssim x\,, \text{ for any } x, y \in A\,. \tag{3}$$

The set π of equivalence classes of $\equiv$ is a partition of A.

Notation. In the following we denote by $[x]_\lesssim$ the equivalence class (the block) of the element x induced by the preorder $\lesssim$ via the equivalence relation defined in (3).

Further, the preorder $\lesssim$ induces on π a partial order $\preccurlyeq$ defined by

$$x \lesssim y \text{ if and only if } [x]_\lesssim \preccurlyeq [y]_\lesssim\,, \text{ for any } x,\, y \in A\,. \tag{4}$$

We call $(\pi, \preccurlyeq)$ *the poset of equivalence classes induced by* $\lesssim$.

This correspondence allows us to give a further definition of monotone and regular partition of a poset by means of preorders.

Definition 2.8 (Monotone partition): A *monotone partition* of a poset $(P, \leqslant)$ is the poset of equivalence classes induced by a preorder $\lesssim$ on P such that $\leqslant \subseteq \lesssim$.

Definition 2.9 (Regular partition): A *regular partition* of a poset $(P, \leqslant)$ is the poset of equivalence classes induced by a preorder $\lesssim$ on P such that $\leqslant \subseteq \lesssim$, and satisfying

$$\lesssim \; = \; \mathrm{tr}\,(\lesssim \setminus \rho)\,, \tag{5}$$

where $\mathrm{tr}\,(R)$ denotes the transitive closure of the relation R, and ρ is a binary relation defined by

$$\rho = \{(x, y) \in P \times P \mid x \lesssim y,\, x \nleqslant y,\, y \not\lesssim x\}\,.$$

The following hold.

Theorem 2.10: Definitions 2.4, 2.6, and 2.8 are equivalent.

Proof: See [2, Theorems 4.1 and 4.4]. ■

Theorem 2.11: Definitions 2.5, 2.7, and 2.9 are equivalent.

Proof: See [2, Theorems 4.2 and 4.5]. ■

The equivalence of the three different definitions for monotone and regular partitions proved in Theorems 2.10 and

2.11, respectively, allows us tu use the more convenient definition depending on the situation. In particular:

- Definitions 2.4 and 2.5 refer to sets of fibres of some special morphisms between posets, generalising thus the definition of (set) partition in terms of fibres of a surjection;
- Definitions 2.6 and 2.7 refer to a partially ordered set of blocks;
- Definitions 2.8 and 2.9 refer to preorders, which can be seen as generalisation of equivalence relations.

III. Indiscernibility Relations on Partially Ordered Sets

Let $\mathcal{A} = (U, A)$ be a pair of nonempty finite sets, where U is the collection of objects (the *universe*), and A a set of *attributes*. Here, attributes are functions $a : U \to V_a$, where V_a is a set of values of attribute a, the *domain* of A. $\mathcal{A}$ is usually called an *information system* (see, *e.g.*, [14]), and can be represented by a table reporting for each object the values of its attributes.

Suppose that a subset of attributes $O = \{a_1, \ldots, a_n\} \subseteq A$ provides a natural way to endow U with a partial order $\leqslant$ defined by

$$x \leqslant y \text{ if and only if } a_i(x) \leqslant a_i(y),$$

for each $i \in \{1, \ldots, n\}$, and for each $x, y \in U$. In such situation, we may want to consider the universe U as a poset, and to reduce the set of attributes to those not involved in the partial order definition. Let then $\bar{A} = A \setminus O$, and let $P = (U, \leqslant)$. We shall denote by $\mathcal{P} = (P, \bar{A})$ the information system having as universe the finite poset P.

As in the 'classical' rough set theory, with a subset of attributes $B \subseteq \bar{A}$ we associate an *indiscernibility relation* on the underlying set of P, denoted by I_B and defined by

$$(x, y) \in I_B \text{ if and only if } a(x) = a(y)\,,$$

for each $a \in \bar{A}$, and for each $x, y \in P$. We will also write $x\ I_B\ y$ in alternative to $(x, y) \in I_B$. Clearly, I_B is an equivalence relation on the underlying set U of P, and thus induces on U a partition $\pi = U/I_B$. We can look at the relation I_B as a way to express the fact that we are unable to observe (to distinguish) individual objects, but we are forced, instead, to think in terms of *granules*, *i.e.*, in terms of blocks of a partition (see, *e.g.*, [12], [18], [20]). In symbols, if $x, y \in P$, x is distinguishable from y if and only if $[x]_\pi \neq [y]_\pi$.

Clearly, the partition π has no reason to be the underlying set of a partition of P in the sense of Definitions 2.6 or 2.7. When it is the case, we say that π is compatible with the the poset, as stated in the following definition.

Definition 3.1: Let $P = (U, \leqslant)$ be a poset, and let π be a partition of U. We say π is *compatible* with P if there exists a monotone partition $(\pi, \leqslant)$ of P. Further, if π is compatible with P we say that π admits an *extension* to a monotone partition of P.

Definition 3.1 naturally extends to a notion of compatibility of the indiscernibility relation I_B: we say I_B is *compatible* with P if and only if $\pi = U/I_B$ is compatible with P.

The question arises, when the partition π is *compatible* with the poset P. In other words, under what conditions π can be *extended* to a monotone or regular partition of P, by endowing π with a partial order relation $\leqslant$. The answer is given by the following.

Corollary 3.2 (Compatibility Criterion): Let $P = (U, \leqslant)$ be a poset, and let π be a partition of U. Then, π is compatible with P if and only if, for all $x, y \in P$,

$$x \lesssim_\pi y \text{ and } y \lesssim_\pi x \text{ imply } [x]_\pi = [y]_\pi\,. \tag{6}$$

Proof: ($\Rightarrow$) Let $(\pi, \leqslant)$ be an extension of π to a monotone partition of P. By Theorem 2.10 we can construct an order-preserving surjection $f : P \to \pi$ such that the set π_f of the fibres of f coincides with π. Let $x \lesssim_\pi y$ and $y \lesssim_\pi x$, for some $x, y \in P$. Then, there exist two sequences $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y$ and $y = z_0, w_0, \ldots, z_m, w_m = x$ satisfying Conditions (1) and (2) in Definition 2.1 with respect to π. Since f is order-preserving, we have $[x]_\pi = f(x) = f(x_0) = f(y_0) \leqslant f(x_1) = f(y_1) \leqslant \cdots \leqslant f(x_n) = f(y_n) = f(y) = [y]_\pi$, and $[y]_\pi = f(y) = f(z_0) = f(w_0) \leqslant f(z_1) = f(w_1) \leqslant \cdots \leqslant f(z_m) = f(w_m) = f(x) = [x]_\pi$.
Thus, $[x]_\pi = [y]_\pi$.

($\Leftarrow$) Let π be a set partition of P such that for all $x, y \in P$, $x \lesssim_\pi y$ and $y \lesssim_\pi x$ imply $[x]_\pi = [y]_\pi$. Define the binary relation $\leqslant$ on π by prescribing that for all $x, y \in P$, $x \lesssim_\pi y$ if and only if $[x]_\pi \leqslant [y]_\pi$. It is immediate to check that $\leqslant$ is a partial order. By Theorem 2.11, $(\pi, \leqslant)$ is a monotone partition – in fact, a regular one. ■

For regular partitions one can say more: a set partition of P admits at most one extension to a regular partition of the poset P.

Corollary 3.3: Let $P = (U, \leqslant)$ be a poset, and let π be a partition of U. If π is compatible with P, then π admits a unique extension to a regular partition of P.

Proof: Define the binary relation $\leqslant$ on π by prescribing that for all $x, y \in P$, $x \lesssim_\pi y$ if and only if $[x]_\pi \leqslant [y]_\pi$. It is immediate to check that $\leqslant$ is a partial order. By Theorem 2.11, $(\pi, \leqslant)$ is a regular partition. To prove uniqueness, consider an extension of π to a regular partition $(\pi, \leqslant')$ of P. Then, $\leqslant'$ must be such that for each $x, y \in P$, $x \lesssim_\pi y$ if and only if $[x]_\pi \leqslant [y]_\pi$, for else we would violate the necessary condition (ii) in Definition 2.7. ■

The uniqueness property of regular partitions proved in the above corollary does not hold, in general, for monotone partitions; cf. Figure 1, which shows three distinct monotone partitions of a given poset P having the same underlying set.

Example 2: Consider the poset P shown in Figure 3 with the depicted set partition $\pi = \{\{a, x\}, \{b, z\}, \{c, y\}, \{d, w\}\}$. The elements a, d of P are such that $a \lesssim_\pi d$ (witness the

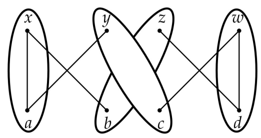

Figure 3. Example 2.

sequence a, a, y, c, w, d) and $d \lesssim_\pi a$ (witness the sequence d, d, z, b, x, a), but a and d are not in the same block. Thus, there is no partition of P having π as its underlying set.

IV. Buying a house: a practical example

Consider the following table, reporting a collection of houses for sale in the city of Merate, Lecco, Italy.

House	Price (€)	Size (m^2)	District	Condition	Rooms
a	200.000	50	Centre	excellent	2
b	170.000	70	Centre	poor	3
c	185.000	53	Centre	very good	2
d	190.000	68	Sartirana	very good	3
e	140.000	60	Sartirana	good	2
f	155.000	65	Novate	good	2
g	250.000	85	Novate	excellent	3
h	240.000	75	Novate	excellent	3

Table I
Attribute-value table.

In this simple information table eight distinct houses are characterised by five attributes: Price, Size, District, Condition, and Rooms. Let $U = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h}\}$ be the set of all houses. We choose the subset of attributes $O = \{\text{Price}, \text{Size}\}$ to define on U a partial order $\leqslant$ as follow. For each $x, y \in U$,

$$x \leqslant y \text{ if and only if } \text{Price}(x) \leqslant \text{Price}(y)\,, \text{Size}(x) \leqslant \text{Size}(y)\,.$$

We obtain the poset $P = (U, \leqslant)$ displayed in Figure 4.

We denote by $\mathcal{P}(P, \bar{A})$ the information system having P as universe, and $\bar{A} = \{\text{District}, \text{Condition}, \text{Rooms}\}$ as the set of attributes. Let $D = \{\text{District}\}$, $C = \{\text{Condition}\}$, and $R = \{\text{Rooms}\}$, and denote by π_D, π_C, and π_R the partitions U/I_D, U/I_C, and U/I_R respectively. Moreover, let $S = D \cup R$, $T = C \cup R$, and let $\pi_S = U/I_S$, and $\pi_T = U/I_T$. We have:

$\pi_D = \{\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}, \{\mathtt{d}, \mathtt{e}\}, \{\mathtt{f}, \mathtt{g}, \mathtt{h}\}\}$;
$\pi_C = \{\{\mathtt{a}, \mathtt{g}, \mathtt{h}\}, \{\mathtt{b}\}, \{\mathtt{c}, \mathtt{d}\}, \{\mathtt{e}, \mathtt{f}\}\}$;
$\pi_R = \{\{\mathtt{a}, \mathtt{c}, \mathtt{e}, \mathtt{f}\}, \{\mathtt{b}, \mathtt{d}, \mathtt{g}, \mathtt{h}\}\}$;
$\pi_S = \{\{\mathtt{a}, \mathtt{c}\}, \{\mathtt{b}\}, \{\mathtt{d}\}, \{\mathtt{e}\}, \{\mathtt{f}\}, \{\mathtt{g}, \mathtt{h}\}\}$;
$\pi_T = \{\{\mathtt{a}\}, \{\mathtt{b}\}, \{\mathtt{c}\}, \{\mathtt{d}\}, \{\mathtt{e}, \mathtt{f}\}, \{\mathtt{g}, \mathtt{h}\}\}$.

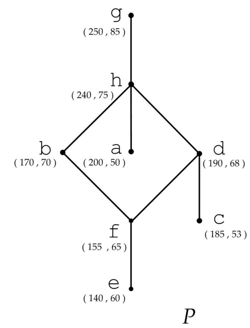

Figure 4. $P = (U, \leqslant)$.

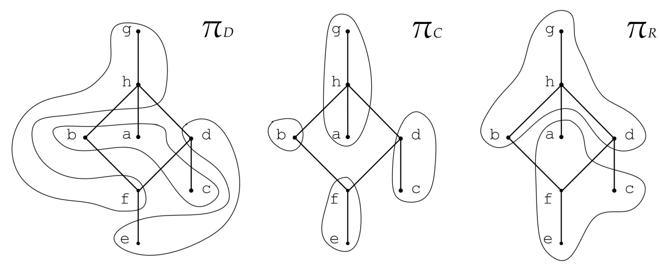

Figure 5. $\pi_D = U/I_D$, $\pi_C = U/I_C$, and $\pi_R = U/I_R$.

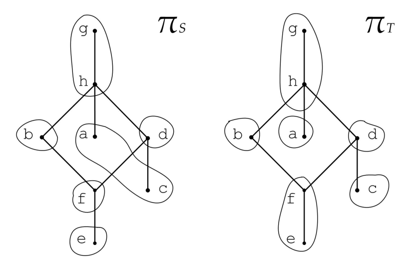

Figure 6. $\pi_S = U/I_{D \cup R}$, and $\pi_T = U/I_{C \cup R}$.

Figure 5 represents on P the partitions π_D, π_C, and π_R. Figure 6 represents on P the partitions π_S, and π_T.

One can easily check, using Corollary 3.2, that while π_C, π_R, π_S, and π_T are compatible with P, and thus can be extended to monotone partitions of P, π_D is not. We can interpret the fact that π_D is not compatible with P as a sort of incoherence (or *incompatibility*) between the indiscernibility relation I_D induced by the attribute District and the partial order $\leqslant$ imposed by the attributes Price and Size. Consider the elements $b, f, h \in P$. The relation I_D tells us that we are unable to make distinction between h and f. But, we can distinguish b from f and h, and thus we are able to 'read' the ordering $f \leqslant b \leqslant h$. In this way, the partial order helps us to discern f from h, in contradiction with what I_D says.

On the other hand, it can be shown that if a partition π of the underlying set of a poset P is compatible with P, and $x, y, z \in P$ are such that $x \leqslant y \leqslant z$ and $[x]_\pi = [z]_\pi$, then $[x]_\pi = [y]_\pi$. (This is, indeed, just a special case of Condition

(6).) Therefore, in case of compatibility, we have a clear way to establish an order, induced by the order of the poset, between blocks of indiscernible elements.

Consider now the partition π_C. The Hasse diagrams in Figure 7 show all the possible ways to extend π_C to a monotone partition of P. The unique extension to a regular partition is shaded.

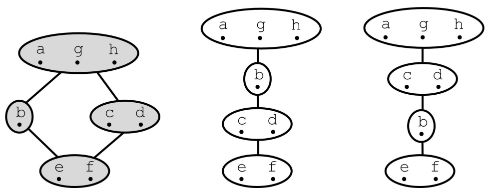

Figure 7. Extensions of π_C.

In some sense, when we consider the (unique) regular partition of a poset P induced by an indiscernibility relation on the underlying set of P compatible with P, we are not adding information to the partial order of P. On the other hand, taking into account monotone partitions which are not regular (see Figure 7, second and third graphics), amounts to add information which was not previously included in the poset. This can be regarded as an attempt to add some *preferences* amongst the elements of the poset. We leave this topic for future research.

V. Conclusion, and Further Research

In this work, we have established a connection between the notions of monotone and regular partitions of a poset P, introduced in Section II, and the notion of indiscernibility relation on the underlying set of P (Section III). Such connection, as proved in Corollary 3.1, can be expressed in terms of *compatibility*.

The investigation of the relations introduced when extending a regular partition to a monotone one, and the study of the structure of all possible indiscernibility relation on a partially ordered universe, are some of the topics of our future research.

References

[1] P. Codara, "A theory of partitions of partially ordered sets," Ph.D. dissertation, Università degli Studi di Milano, Italy, 2008.

[2] ——, "Partitions of a Finite Partially Ordered Set," in *From Combinatorics to Philosophy. The Legacy of G.-C. Rota*, E. Damiani, O. D'Antona, V. Marra, and F. Palombi, Eds. New York: Springer US, 2009, pp. 45–59.

[3] P. Codara, O. M. D'Antona, and V. Marra, "Open Partitions and Probability Assignments in Gödel Logic," in *ECSQARU*, ser. LNCS (LNAI), vol. 5590. Springer, 2009, pp. 911–922.

[4] S. Greco, B. Matarazzo, and R. Slowinski, "Rough sets theory for multicriteria decision analysis," *European Journal of Operational Research*, vol. 129, no. 1, pp. 1–47, 2001.

[5] X. Hu and N. Cercone, "Learning in relational databases: A rough set approach," *Computational Intelligence*, vol. 11, pp. 323–338, 1995.

[6] X. Hu, T. Y. Lin, and J. Han, "A new rough sets model based on database systems," *Fundam. Inf.*, vol. 59, pp. 135–152, July 2003.

[7] J. Jelonek, K. Krawiec, and R. Slowinski, "Rough set reduction of attributes and their domains for neural networks," *Computational Intelligence*, vol. 11, pp. 339–347, 1995.

[8] M. Kryszkiewicz, "Rough set approach to incomplete information systems," *Inform. Sci.*, vol. 112, no. 1-4, pp. 39–49, 1998.

[9] G. Liu and W. Zhu, "The algebraic structures of generalized rough set theory," *Inform. Sci.*, vol. 178, no. 21, pp. 4105–4113, 2008.

[10] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 1998, vol. 5.

[11] O. Ore, "Theory of equivalence relations," *Duke Math. J.*, vol. 9, pp. 573–627, 1942.

[12] E. Orłowska, Ed., *Incomplete information: rough set analysis*, ser. Studies in Fuzziness and Soft Computing. Heidelberg: Physica-Verlag, 1998, vol. 13.

[13] P. Pagliani and M. Chakraborty, *A geometry of approximation*, ser. Trends in Logic—Studia Logica Library. New York: Springer, 2008, vol. 27.

[14] Z. Pawlak, "Information systems - theoretical foundations," *Inf. Syst.*, vol. 6, no. 3, pp. 205–218, 1981.

[15] ——, "Rough sets," *Internat. J. Comput. Inform. Sci.*, vol. 11, no. 5, pp. 341–356, 1982.

[16] ——, "Rough set approach to knowledge-based decision support," *European Journal of Operational Research*, vol. 99, pp. 48–57, 1995.

[17] Z. Pawlak and A. Skowron, "Rough sets and Boolean reasoning," *Inform. Sci.*, vol. 177, no. 1, pp. 41–73, 2007.

[18] ——, "Rudiments of rough sets," *Inf. Sci.*, vol. 177, no. 1, pp. 3–27, 2007.

[19] Y. Y. Yao, "Two views of the theory of rough sets in finite universes," *International Journal of Approximate Reasoning*, vol. 15, pp. 291–317, 1996.

[20] ——, "Information granulation and rough set approximation," *Int. J. Intell. Syst.*, vol. 16, no. 1, pp. 87–104, 2001.

[21] Y. Y. Yao and T. Y. Lin, "Generalization of rough sets using modal logics," *Intelligent Automation and Soft Computing, an International Journal*, vol. 2, pp. 103–120, 1996.