

Valuations in Gödel Logic, and the Euler Characteristic

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Using the lattice-theoretic version of the Euler characteristic introduced by V. Klee and G.-C. Rota in the Sixties, we define the Euler characteristic of a formula in Gödel logic (over finitely or infinitely many truth-values). We then prove that the information encoded by the Euler characteristic is classical, i.e. coincides with the analogous notion defined over Boolean logic. Building on this, we define many-valued versions of the Euler characteristic of a formula φ , and prove that they indeed provide information about the logical status of φ in Gödel logic. Specifically, our first main result shows that the many-valued Euler characteristics are invariants that separate many-valued tautologies from non-tautologies. Further, we offer an initial investigation of the linear structure of these generalised characteristics. Our second main result is that the collection of many-valued characteristics forms a linearly independent set in the real vector space of all valuations of Gödel logic over finitely many propositional variables.

Key words: Gödel logic, Gödel algebra, distributive lattice, Euler characteristic, valuation, vector space of valuations.

1 INTRODUCTION AND BACKGROUND

Some decades ago, V. Klee and G.-C. Rota introduced a lattice-theoretic analogue of the Euler characteristic, the celebrated topological invariant of polyhedra. Let us recall their definition. Let L be a distributive lattice. A function

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$\nu: L \rightarrow \mathbb{R}$ is a *valuation* if it satisfies

$$\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y) \quad (1)$$

for all $x, y, z \in L$. Recall that an element $x \in L$ is *join-irreducible* if it is not the bottom element of L , and $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$. When L is finite, it turns out [13, Corollary 2] that any valuation ν is uniquely determined by its values on the join-irreducible elements of L , along with its value at the bottom element \perp of L .

Definition 1.1 ([11, p. 120], [13, p. 36]). *The Euler characteristic of a finite distributive lattice L is the unique valuation $\chi: L \rightarrow \mathbb{R}$ such that $\chi(x) = 1$ for any join-irreducible element $x \in L$, and $\chi(\perp) = 0$.*

Gödel (*infinite-valued propositional*) logic \mathbb{G}_∞ [7] can be syntactically defined as the schematic extension of the intuitionistic propositional calculus by the *prelinearity axiom* $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$. It can also be semantically defined as a many-valued logic [8], as follows. Write FORM for the set of formulæ over propositional variables X_1, X_2, \dots in the language $\wedge, \vee, \rightarrow, \neg, \perp, \top$. (Here, \perp and \top are the logical constants *falsum* and *verum*, respectively.) An *assignment* is a function $\mu: \text{FORM} \rightarrow [0, 1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha, \beta \in \text{FORM}$,

$$\begin{aligned} \mu(\alpha \wedge \beta) &= \min\{\mu(\alpha), \mu(\beta)\} \\ \mu(\alpha \vee \beta) &= \max\{\mu(\alpha), \mu(\beta)\} \\ \mu(\alpha \rightarrow \beta) &= \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases} \end{aligned}$$

and $\mu(\neg\alpha) = \mu(\alpha \rightarrow \perp)$, $\mu(\perp) = 0$, $\mu(\top) = 1$. A *tautology* is a formula α such that $\mu(\alpha) = 1$ for every assignment μ . As is well known, Gödel logic is complete with respect to this many-valued semantics. Indeed, for $\alpha \in \text{FORM}$, let us write $\vdash \alpha$ to mean that α is derivable from the axioms of \mathbb{G}_∞ using *modus ponens* as the only deduction rule. Then the completeness theorem guarantees that $\vdash \alpha$ holds if and only if α is a tautology. A stronger result holds: like classical logic, \mathbb{G}_∞ also enjoys completeness for theories. For proofs and background, see [8].

For an integer $n \geq 1$, let us write FORM_n for the set of all formulæ whose propositional variables are contained in $\{X_1, \dots, X_n\}$. As usual, $\varphi, \psi \in \text{FORM}_n$ are called *logically equivalent* if both $\vdash \varphi \rightarrow \psi$ and $\vdash \psi \rightarrow \varphi$ hold.

Logical equivalence is an equivalence relation, written \equiv , and its equivalence classes are denoted $[\varphi]_{\equiv}$. By a routine check, the quotient set FORM_n / \equiv endowed with operations $\wedge, \vee, \top, \perp$ induced from the corresponding logical connectives becomes a distributive lattice with top and bottom element \top and \perp , respectively. When FORM_n / \equiv is further endowed with the operation \rightarrow induced by implication, it becomes a Heyting algebra satisfying prelinearity; such algebras we call *Gödel algebras* (cf. the term *G-algebras* in [8, 4.2.12]). The specific Gödel algebra $\mathcal{G}_n = \text{FORM}_n / \equiv$ is, by construction, the *Lindenbaum algebra* of Gödel logic over the language $\{X_1, \dots, X_n\}$.

It is a remarkable fact due to Horn [9, Theorem 4] that \mathcal{G}_n is finite for each integer $n \geq 1$, in analogy with Boolean algebras. A second important fact is that a finite Heyting algebra is a Gödel algebra if and only if its collection of join-irreducible elements, ordered by restriction from \mathcal{G}_n , is a *forest*; i.e. the lower bounds of any such element are a totally ordered set. A more general version of this result is also due to Horn [10, Theorem 2.4].

Knowing that \mathcal{G}_n is a finite distributive lattice whose elements are formulæ in n variables, up to logical equivalence, one is led to give the following definition.

Definition 1.2. *The Euler characteristic of a formula $\varphi \in \text{FORM}_n$, written*

$$\chi(\varphi),$$

is the number $\chi([\varphi]_{\equiv})$, where χ is the Euler characteristic of the finite distributive lattice \mathcal{G}_n .

However, the question is now whether $\chi(\varphi)$ encodes genuinely *logical* information about φ , just like the Euler characteristic of a polyhedron provides geometric information about that polyhedron. The answer turns out to be affirmative. As usual, we say that an assignment $\mu: \text{FORM}_n \rightarrow [0, 1]$ is *Boolean* if it takes values in $\{0, 1\}$.

Theorem 1.3. *Fix an integer $n \geq 1$. For any formula $\varphi \in \text{FORM}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu: \text{FORM}_n \rightarrow \{0, 1\}$ such that $\mu(\varphi) = 1$.*

Theorem 1.3 will turn out to be an easy corollary of our first main result, Theorem 2.3. As an immediate consequence of Theorem 1.3,

$$0 \leq \chi(\varphi) \leq 2^n$$

for any $\varphi \in \text{FORM}_n$. In particular, note that the following hold.

- If φ is a tautology in \mathbb{G}_∞ , then $\chi(\varphi) = 2^n$.
- If $\chi(\varphi) = 2^n$, then φ is a tautology in classical propositional logic.
- If $\chi(\varphi) = 0$, then φ is a contradiction in classical propositional logic, and conversely.

In summary, Theorem 1.3 shows that, while $\chi(\varphi)$ does encode non-trivial logical information, that information is classical, and independent of Gödel logic. In fact, if one replicates the above construction over classical logic, one ends up with a valuation χ on the Boolean algebra of n -variable formulæ that simply counts the number of atoms below each element in the Boolean algebra. By the same token, the Euler characteristic cannot tell apart the tautologies in Gödel logic from the remaining formulæ, whereas it does so for classical tautologies. In Section 2 we show how to remedy this by considering different valuations on \mathcal{G}_n which we refer to as *generalised characteristics* (Definition 2.1). As it will emerge, they can be thought of as many-valued variants of the classical characteristic of Definition 1.2.

Our first main result, Theorem 2.3, shows that χ_k is a natural generalisation of χ in that it tells apart the tautologies in *Gödel $(k+1)$ -valued logic* \mathbb{G}_{k+1} from the remaining formulæ. Here we recall that \mathbb{G}_{k+1} is the schematic extension of \mathbb{G}_∞ via

$$\alpha_1 \vee (\alpha_1 \rightarrow \alpha_2) \vee \cdots \vee (\alpha_1 \wedge \cdots \wedge \alpha_k \rightarrow \alpha_{k+1}). \quad (2)$$

Alternatively, using [4, Proposition 4.18], one can equivalently replace* (2) by the axiom

$$\bigvee_{1 \leq i \leq k} (\alpha_i \rightarrow \alpha_{i+1}).$$

Semantically, restrict assignments to those taking values in the set

$$V_{k+1} = \left\{0 = \frac{0}{k}, \frac{1}{k}, \dots, \frac{k-1}{k}, \frac{k}{k} = 1\right\} \subseteq [0, 1],$$

that is, to *$(k+1)$ -valued assignments*. A tautology of \mathbb{G}_{k+1} is defined as a formula that takes value 1 under any such assignment. Then \mathbb{G}_{k+1} is complete with respect to this semantics; see e.g. [3] for further background.

In Section 3, we analyse the linear structure of the generalised characteristics introduced in Section 2. The set of valuations over a finite distributive lattice L carries a natural structure of (real) vector space. This is because the

* We thanks the anonymous referee for bringing [4] to our attention.

function $\zeta: L \rightarrow \mathbb{R}$ such that $\zeta(x) = 0$ for each $x \in L$ is a valuation, and if $\nu_1, \nu_2: L \rightarrow \mathbb{R}$ are valuations, then so is the function $r_1\nu_1 + r_2\nu_2$ defined by

$$(r_1\nu_1 + r_2\nu_2)(x) = r_1\nu_1(x) + r_2\nu_2(x) \text{ for each } x \in L,$$

for any two real numbers $r_1, r_2 \in \mathbb{R}$. It is therefore natural to ask what linear relations are satisfied by the generalised characteristics. As we prove in our second main result (Theorem 3.3) the answer is none.

2 THE MANY-VALUED CHARACTERISTIC OF A FORMULA

The *height* of a join-irreducible $g \in \mathcal{G}_n$ is the length l of the longest chain $g = g_1 > g_2 > \dots > g_l$ in \mathcal{G}_n with each g_i a join-irreducible element. We write $h(g)$ for the height of g .

We can now define the generalised characteristics that feature in Section 1.

Definition 2.1. Fix integers $n, k \geq 1$. We write $\chi_k: \mathcal{G}_n \rightarrow \mathbb{R}$ for the unique valuation on \mathcal{G}_n that satisfies

$$\chi_k(g) = \min \{h(g), k\}$$

for each join-irreducible element $g \in \mathcal{G}_n$, and such that, moreover, $\chi_k(\perp) = 0$. Further, if $\varphi \in \text{FORM}_n$, we define $\chi_k(\varphi) = \chi_k([\varphi]_{\equiv})$.

Clearly, χ_1 is the Euler characteristic χ of \mathcal{G}_n . We now need to recall a notion (cfr. [5, Definition 2.1]) that is central to Gödel logic.

Definition 2.2. Fix integers $n, k \geq 1$. We say that two $(k+1)$ -valued assignments μ and ν are equivalent over the first n variables, or n -equivalent, written $\mu \equiv_n^k \nu$, if and only if there exists a permutation $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that

$$\begin{aligned} 0 \preceq_0 \mu(X_{\sigma(1)}) \preceq_1 \dots \preceq_{n-1} \mu(X_{\sigma(n)}) \preceq_n 1, \\ 0 \preceq_0 \nu(X_{\sigma(1)}) \preceq_1 \dots \preceq_{n-1} \nu(X_{\sigma(n)}) \preceq_n 1, \end{aligned} \tag{3}$$

where $\preceq_i \in \{<, =\}$, for $i = 0, \dots, n$.

Thus, two equivalent assignments induce the same strict inequalities ($<$) and equalities ($=$) on the propositional variables. Clearly, \equiv_n^k is an equivalence relation. In various guises, the above notion of equivalent assignments

plays a crucial rôle in the investigation of Gödel logic; see e.g. [5, 2]. For our purposes here, we observe that distinct 2-valued (=Boolean) assignments are not equivalent, so that there are 2^n equivalence classes of such assignments over the first n variables.

We next introduce the $(k + 1)$ -valued analogue of 2^n . As will be proved in Subsection 2.1, the following recursive formula counts the number of join-irreducible elements of \mathcal{G}_n having height smaller or equal than k .

$$P(n, k) = \sum_{i=1}^k \sum_{j=0}^n \binom{n}{j} T(j, i), \quad (*)$$

where

$$T(n, k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > n + 1, \\ \sum_{i=1}^n \binom{n}{i} T(n - i, k - 1) & \text{otherwise.} \end{cases}$$

	k=1	2	3	4	5	6	7
n=1	2	3	3	3	3	3	3
2	4	9	11	11	11	11	11
3	8	27	45	51	51	51	51
4	16	81	191	275	299	299	299
5	32	243	813	1563	2043	2163	2163
6	64	729	3431	8891	14771	18011	18731
7	128	2187	14325	49731	106851	158931	184131
8	256	6561	59231	272675	757019	1407179	1921259
9	512	19683	242973	1468203	5228043	12200883	20214483

TABLE 1
The number of distinct equivalence classes of $(k + 1)$ -valued assignments over n variables.

Our aim in this section is to establish the following result.

Theorem 2.3. Fix integers $n, k \geq 1$, and a formula $\varphi \in \text{FORM}_n$.

1. $\chi_k(\varphi)$ equals the number of $(k+1)$ -valued assignments $\mu: \text{FORM}_n \rightarrow [0, 1]$ such that $\mu(\varphi) = 1$, up to n -equivalence.
2. φ is a tautology in \mathbb{G}_{k+1} if and only if $\chi_k(\varphi) = P(n, k)$.
3. φ is a tautology in \mathbb{G}_∞ if and only if it is a tautology in \mathbb{G}_{n+2} if and only if $\chi_{n+1}(\varphi) = P(n, n+1)$.

Since distinct Boolean assignments are pairwise inequivalent, Theorem 1.3 is an immediate consequence of Theorem 2.3. We note that

$$P(n, 1) = \sum_{j=0}^n \binom{n}{j} T(j, 1) = \sum_{j=0}^n \binom{n}{j} = 2^n,$$

so that $P(n, k)$ indeed is the $(k+1)$ -valued analogue of 2^n .

Remark. A closed formula for the number $P(n, k)$ may be obtained combining the results of [12] on the number of chains in a power set, and the results of [6, Subsection 4.2] relating the number of join-irreducible elements of \mathcal{G}_n to ordered partitions of finite sets. We do not provide the combinatorial details in the present paper.

2.1 Proof of Theorem 2.3

Proof of ()*

Let \mathcal{F}_n be the forest of join-irreducible elements of \mathcal{G}_n , and let \mathcal{T}_n be the unique tree of \mathcal{F}_n having maximum height (cfr. [2, Section 2.3]). Here, by the height $h(F)$ of a forest F we mean the cardinality of its longest chain. Denote by $\uparrow g$ the upper set of an element g , that is,

$$\uparrow g = \{x \in F \mid x \geq g\}.$$

Similarly, the lower set of g is

$$\downarrow g = \{x \in F \mid x \leq g\}.$$

The height of an element $g \in F$ is the height of $\downarrow g$. Recall that an *atom* of a partially ordered set with minimum is an element that covers its minimum. It can be shown (cfr. [2, Lemma 2.3 – (a)]) that \mathcal{T}_n has precisely $\binom{n}{i}$ atoms a with $\uparrow a \cong \mathcal{T}_{n-i}$, for each $i = 1, \dots, n$, and no other atom. Observing that \mathcal{T}_0 is the one-element tree, and that $h(\mathcal{T}_n) = h(\mathcal{T}_{n-1}) + 1$ for each

n , we immediately obtain the following recursive formula for the number of elements of \mathcal{T}_n having height k .

$$T(n, k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > n + 1, \\ \sum_{i=1}^n \binom{n}{i} T(n-i, k-1) & \text{otherwise.} \end{cases}$$

Further, \mathcal{T}_n contains precisely $\binom{n}{i}$ distinct copies of \mathcal{T}_i , for $i = 0, \dots, n$, and no other tree (cfr. [2, Lemma 2.3 – (b)]). Thus, as claimed, $P(n, k)$ gives the number of elements of \mathcal{T}_n having height smaller or equal than k (i.e. the number of join-irreducible elements of \mathcal{G}_n having height smaller or equal than k).

Two lemmas

Lemma 2.4. *Fix integers $n, k \geq 1$, let $x \in \mathcal{G}_n$ and consider the valuation $\chi_k: \mathcal{G}_n \rightarrow \mathbb{R}$. Then, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq x$ and $h(g) \leq k$.*

Proof. If $x = \perp$ then, by Definition 2.1, $\chi_k(x) = 0$, and the Lemma trivially holds.

Let F be the forest of all join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq x$. (Recall that x is the join of the join-irreducible elements $g \in F$.) We proceed by induction on the structure of F . If F is the one-element forest, then x is a join-irreducible element, and $F = \{x\}$. By Definition 2.1, $\chi_k(x) = 1$, for each $k \geq 1$, as desired.

Let now $|F| > 1$. Let $l \in F$ be a maximal element of F , and consider the forest $F^- = F \setminus \{l\}$. Let x^- be the join of the elements of F^- . We immediately observe that $x = l \vee x^-$.

If l is an atom of \mathcal{G}_n , then $l \wedge x^- = \perp$. By (1) and Definition 2.1, $\chi_k(x) = \chi_k(l \vee x^-) = \chi_k(l) + \chi_k(x^-) - \chi_k(l \wedge x^-) = 1 + \chi_k(x^-)$. Using the inductive hypotheses on F^- we obtain our statement, for the case $h(l) = 1$.

Let, finally, $h(l) > 1$. Consider the element $l^- = l \wedge x^-$. Let L be the forest of all join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq l$, and let L^- be the forest of all join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq l^-$. Since l is a join-irreducible, L is a chain. Moreover, one easily sees that $L^- = L \setminus \{l\}$. For a forest P , we denote by $|P|_k$ the number of elements p of P such that $h(p) \leq k$. We consider two cases.

$h(l) \leq k$. We observe that $|F^-|_k = |F|_k - 1$ and that $|L^-|_k = |L|_k - 1$. Using (1) and the inductive hypotheses we obtain $\chi_k(x) = \chi_k(l) + \chi_k(x^-) - \chi_k(l \wedge x^-) = |L|_k + |F|_k - 1 - (|L|_k - 1) = |F|_k$. In other words, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq x$ and $h(g) \leq k$.

$h(l) > k$. In this case, we observe that $|F^-|_k = |F|_k$ and that $|L^-|_k = |L|_k$. Using (1) and the inductive hypotheses we obtain $\chi_k(x) = \chi_k(l) + \chi_k(x^-) - \chi_k(l \wedge x^-) = |L|_k + |F|_k - |L|_k = |F|_k$. In other words, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq x$ and $h(g) \leq k$, and the lemma is proved. \square

Lemma 2.5. *Fix integers $n, k \geq 1$, and let $\varphi \in \text{FORM}_n$. Let $O(\varphi, n, k)$ be the set of equivalence classes $[\mu]_{\equiv_n^k}$ of $(k+1)$ -valued assignments such that $\mu(\varphi) = 1$. Further, let $J(\varphi, n, k)$ be the set of join-irreducible elements $x \in \mathcal{G}_n$ such that $x \leq [\varphi]_{\equiv}$ and $h(x) \leq k$. Then there is a bijection between $O(\varphi, n, k)$ and $J(\varphi, n, k)$.*

Proof. In the proof of this lemma we use techniques from algebraic logic; for all unexplained notions, please see [8].

Fix a $(k+1)$ -valued assignment $\mu: \text{FORM}_n \rightarrow V_{k+1}$. Endow V_{k+1} with its unique structure of Gödel algebra compatible with the total order of the elements of $V_{k+1} \subseteq [0, 1]$. Then there is a unique homomorphism of Gödel algebras $h_\mu: \mathcal{G}_n \rightarrow V_{k+1}$ corresponding to μ , namely,

$$h_\mu([\varphi]_{\equiv}) = \mu(\varphi). \quad (4)$$

Conversely, given any such homomorphism $h: \mathcal{G}_n \rightarrow V_{k+1}$, there is a unique $(k+1)$ -valued assignment $\mu_h: \text{FORM}_n \rightarrow V_{k+1}$ corresponding to h , namely,

$$\mu_h(\varphi) = h([\varphi]_{\equiv}). \quad (5)$$

Clearly, the correspondences (4–5) are mutually inverse, and thus yield a bijection between $(k+1)$ -valued assignments to FORM and homomorphisms $\mathcal{G}_n \rightarrow V_{k+1}$. Further, upon noting that $\mu_h(\varphi) = 1$ in (5) if and only if $h_\mu([\varphi]_{\equiv}) = 1$ in (4), we see that this bijection restricts to a bijection

$$O'(\varphi, n, k) \cong \text{hom}(\varphi, \mathcal{G}_n, V_{k+1}) \quad (6)$$

where the right-hand side is the set of homomorphisms $h: \mathcal{G}_n \rightarrow V_{k+1}$ such that $h([\varphi]_{\equiv}) = 1$, while the left-hand side is the collection of $(k+1)$ -valued

assignments $\mu: \text{FORM}_n \rightarrow V_{k+1}$ with $\mu(\varphi) = 1$. Now recall that to any homomorphism $h: \mathcal{G}_n \rightarrow V_{k+1}$ one associates the prime (lattice) filter of \mathcal{G}_n given by $\mathfrak{p}_h = h^{-1}(1)$. Conversely, given a prime filter \mathfrak{p} of \mathcal{G}_n there is a natural onto quotient map $h_{\mathfrak{p}}: \mathcal{G}_n \twoheadrightarrow \mathcal{G}_n/\mathfrak{p}$, where $C = \mathcal{G}_n/\mathfrak{p}$ is a chain of finite cardinality; further, $|C| \leq k+1$ if and only if \mathfrak{p} has height $\leq k$, meaning that the chain of prime filters containing it has cardinality k . Since any chain with $|C| \leq k+1$ embeds into V_{k+1} , this shows that each prime filter \mathfrak{p} of \mathcal{G}_n having height $\leq k$ induces by

$$h_{\mathfrak{p}}^e: \mathcal{G}_n \twoheadrightarrow \mathcal{G}_n/\mathfrak{p} \xrightarrow{e} V_{k+1} \quad (7)$$

one homomorphism (not necessarily onto) $h_{\mathfrak{p}}^e$ from \mathcal{G}_n to V_{k+1} for *each* choice of the embedding $e: \mathcal{G}_n/\mathfrak{p} \hookrightarrow V_{k+1}$. It is now easy to check that two $(k+1)$ -valued assignments $\mu, \nu: \text{FORM}_n \rightarrow V_{k+1}$ satisfy $\mu \equiv_n^k \nu$ if and only if the associated homomorphisms h_{μ}, h_{ν} as in (4) factor as in (7) for the *same* prime filter \mathfrak{p} , although for possibly different embeddings e and e' into V_{k+1} . It is clear that this yields an equivalence relation on such homomorphisms h_{μ}, h_{ν} . Let us denote by $\text{hom}_{\equiv}(\varphi, \mathcal{G}_n, V_{k+1})$ the set of equivalence classes of those homomorphisms h_{μ} satisfying $h_{\mu}([\varphi]_{\equiv}) = 1$. Summing up, from the bijection in (6) we obtain a bijection

$$O(\varphi, n, k) \cong \text{hom}_{\equiv}(\varphi, \mathcal{G}_n, V_{k+1}) . \quad (8)$$

To complete the proof, observe that since \mathcal{G}_n is finite, every filter \mathfrak{p} of \mathcal{G}_n is principal, i.e. if there is an element $p \in \mathcal{G}_n$ such that $\mathfrak{p} = \uparrow p$; moreover, \mathfrak{p} is prime if and only if p is join-irreducible. In other words, there is a bijection between join-irreducible elements and prime filters of \mathcal{G}_n . By definition, the natural quotient map $\mathcal{G}_n \twoheadrightarrow \mathcal{G}_n/\mathfrak{p}$ sends $[\varphi]_{\equiv}$ to 1 if and only if $[\varphi]_{\equiv}$ lies in the prime filter \mathfrak{p} ; that is, if and only if $[\varphi]_{\equiv} \geq p$ in \mathcal{G}_n . Moreover, the following is easily checked. Suppose $\mathfrak{p} = \uparrow p$ as in the above, and let $\mathcal{G}_n/\uparrow p$ be the quotient algebra, which is a chain because \mathfrak{p} is prime. Then $|\mathcal{G}_n/\uparrow p| \leq k+1$ if and only if the height of p satisfies $h(p) \leq k$. Using the preceding observations, from (7) and the definition of $\text{hom}_{\equiv}(\varphi, \mathcal{G}_n, V_{k+1})$ we obtain a bijection

$$\text{hom}_{\equiv}(\varphi, \mathcal{G}_n, V_{k+1}) \cong J(\varphi, n, k) . \quad (9)$$

The lemma follows from (8) and (9). \square

2.2 End of Proof of Theorem 2.3

1. By Lemma 2.4 the value $\chi_k(\varphi) = \chi_k([\varphi]_{\equiv})$ is given by the number of join-irreducible elements $g \in \mathcal{G}_n$ such that $g \leq [\varphi]_{\equiv}$ and $h(g) \leq k$.

By Lemma 2.5, such number equals the number of equivalence classes $[\mu]_{\equiv_n^k}$ of $(k+1)$ -valued assignments such that $\mu(\varphi) = 1$, and the statement follows.

2. As proved in Subsection 2.1, the formula $P(n, k)$ counts the total number of join-irreducible elements of \mathcal{G}_n having height smaller or equal than k . By Lemma 2.4, $\chi_k(\varphi) = P(n, k)$ if and only if all the join-irreducible elements $g \in \mathcal{G}_n$ such that $h(g) \leq k$ satisfy $g \leq [\varphi]_{\equiv}$. By Lemma 2.5, the latter holds if and only if each $(k+1)$ -valued assignment $\mu: \text{FORM}_n \rightarrow [0, 1]$ satisfies $\mu(\varphi) = 1$, i.e. φ is a tautology in \mathbb{G}_{k+1} , as desired.
3. *Claim:* If $\varphi \in \text{FORM}_n$ is a tautology in \mathbb{G}_{n+2} , then it is a tautology in \mathbb{G}_{∞} .

Proof of Claim: Suppose, by way of contradiction, that φ is not a tautology in \mathbb{G}_{∞} , but it is a tautology in \mathbb{G}_{n+2} . Thus, there must exist an assignment μ such that $\mu(\varphi) < 1$. An easy structural induction shows that $\mu(\varphi) \in \{0, \mu(X_1), \dots, \mu(X_n), 1\}$. But then, the restriction of μ onto its image yields an $(n+2)$ -valued assignment $\bar{\mu}$ such that $\bar{\mu}(\varphi) < 1$, a contradiction.

As one can immediately check, if φ is a tautology in \mathbb{G}_{∞} , then it is a tautology in \mathbb{G}_{n+2} . Thus, using the Claim, φ is a tautology in \mathbb{G}_{n+2} if and only if it is a tautology in \mathbb{G}_{∞} . Finally, by statement 2) of this theorem, φ is a tautology in \mathbb{G}_{n+2} if and only if $\chi_{n+1}(\varphi) = P(n, n+1)$, and the last statement of the theorem is proved.

Example 1. Let us consider the Gödel algebra \mathcal{G}_1 , depicted in Figure 1. Lemma 2.4 allows us to compute the values of $\chi_k(x)$ for each $x \in \mathcal{G}_1$, simply by counting the number of join-irreducible elements under x having height not greater than k . The results are displayed in Figure 1, for $k = 1$ (i.e. for the Euler characteristic), and for $k = 2$. Note that for $k \geq 3$ and for each $x \in \mathcal{G}_1$, $\chi_k(x)$ and $\chi_2(x)$ coincide, by statement 3 in Theorem 2.3.

Let us consider the formula $\neg\neg X$. One can check that, up to n -equivalence, there are two distinct 3-valued assignments $\mu, \nu: \text{FORM}_1 \rightarrow \{0, \frac{1}{2}, 1\}$ such that $\mu(\neg\neg X) = \nu(\neg\neg X) = 1$. Namely, we can take μ such that $\mu(X) = 1$, and ν such that $\nu(X) = \frac{1}{2}$. In fact, as one sees in Figure 1, $\chi_2(\neg\neg X) = 2$. The assignment $\mu(X)$ is the only Boolean assignment such that $\mu(\neg\neg X) = 1$. Actually, $\chi_1(\neg\neg X) = \chi(\neg\neg X) = 1$.

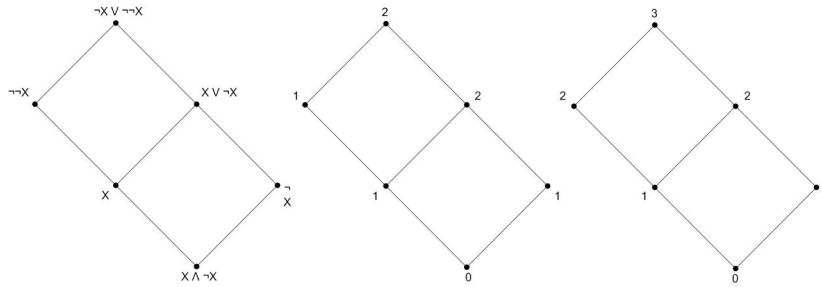


FIGURE 1
The Gödel algebra \mathcal{G}_1 (left), and the values of χ_1 (middle) and χ_2 (right).

3 THE LINEAR STRUCTURE OF THE CHARACTERISTICS

Let J_1, \dots, J_{u_n} display all join-irreducible elements of \mathcal{G}_n , for an integer $n \geq 1$. For $i = 1, \dots, u_n$, let e_i be the unique valuation of \mathcal{G}_n such that $e_i(J_i) = 1$, and $e_i(J_j) = 0$ if $j \neq i$. A moment's reflection shows that $\{e_1, \dots, e_{u_n}\}$ is a basis of the vector space of all valuations of \mathcal{G}_n . Hence, u_n is the dimension of this space. Let us remark that it follows from the proof of (*) in Subsection 2.1 that

$$u_n = P(n, n+1).$$

An *automorphism* of \mathcal{G}_n is a bijective homomorphism of distributive lattices $\alpha: \mathcal{G}_n \rightarrow \mathcal{G}_n$. Such homomorphism is then automatically a homomorphism of Heyting (a fortiori Gödel) algebras, too. A valuation $\nu: \mathcal{G}_n \rightarrow \mathcal{G}_n$ is *invariant* (under the automorphisms of \mathcal{G}_n) if

$$\nu(x) = \nu(\alpha(x)) \text{ for all } x \in \mathcal{G}_n,$$

where α is an arbitrary automorphism of \mathcal{G}_n . The invariant valuations of any finite distributive lattice form a vector subspace of the vector space of all valuations, as one checks easily.

Definition 3.1. We denote by \mathcal{V}_n the vector space of all valuations of \mathcal{G}_n , for an integer $n \geq 1$. We further denote by \mathcal{I}_n the vector subspace of \mathcal{V}_n consisting of all invariant valuations of \mathcal{G}_n . Finally, we write \mathcal{C}_n for the vector subspace of \mathcal{V}_n generated by the generalised characteristics $\{\chi_1, \chi_2, \dots, \chi_{n+1}\}$.

By definition, then, $\mathcal{I}_n, \mathcal{C}_n \subseteq \mathcal{V}_n$. More is true.

Proposition 3.2. *For each integer $n \geq 2$, $\mathcal{C}_n \subset \mathcal{I}_n \subset \mathcal{V}_n$. Further, $\mathcal{C}_1 \subset \mathcal{I}_1 = \mathcal{V}_1$.*

Proof. Let us exhibit a non-invariant valuation of \mathcal{G}_n for each $n \geq 2$. Consider the formulæ

$$\begin{aligned}\varphi &= X_1 \wedge \neg X_2 \wedge \neg X_3 \wedge \cdots \wedge \neg X_n, \\ \psi &= \neg X_1 \wedge X_2 \wedge \neg X_3 \wedge \cdots \wedge \neg X_n.\end{aligned}$$

It can be checked that $[\varphi]_{\equiv}$ and $[\psi]_{\equiv}$ are join-irreducible elements (in fact, atoms) of \mathcal{G}_n . Consider the valuation $\nu: \mathcal{G}_n \rightarrow \mathbb{R}$ such that $\nu([\varphi]_{\equiv}) = 1$, while $\nu(x) = 0$ for every other join-irreducible $x \in \mathcal{G}_n$. The permutation

$$\begin{aligned}X_1 &\mapsto X_2, \\ X_2 &\mapsto X_1, \\ X_i &\mapsto X_i \text{ for each } i = 3, \dots, n\end{aligned}$$

uniquely extends to an automorphism α of \mathcal{G}_n . By construction, $\alpha([\varphi]_{\equiv}) = [\psi]_{\equiv}$. But then $\nu([\varphi]_{\equiv}) = 1 \neq 0 = \nu([\psi]_{\equiv}) = \nu(\alpha([\varphi]_{\equiv}))$. This shows that $\mathcal{I}_n \subset \mathcal{V}_n$ when $n \geq 2$. On the other hand, direct inspection of \mathcal{G}_1 (cf. Example 1) shows that the automorphism group of \mathcal{G}_1 is trivial, i.e. it consists of the identity function only. Hence, $\mathcal{I}_1 = \mathcal{V}_1$.

Finally, we prove $\mathcal{C}_n \subset \mathcal{I}_n$ for each $n \geq 1$. Consider the formula $\gamma = \neg X_1 \wedge \cdots \wedge \neg X_n$. It is easily seen that $J = [\gamma]_{\equiv}$ is a join-irreducible element of \mathcal{G}_n . Moreover, every automorphism of \mathcal{G}_n must fix J . To see this, one checks that J is the only element of \mathcal{G}_n such that (i) J has height 1, and (ii) no join-irreducible element of \mathcal{G}_n is greater than J . Since any automorphism of \mathcal{G}_n must preserve properties (i) and (ii) of J , it follows that every such automorphism fixes J . The valuation $\nu: \mathcal{G}_n \rightarrow \mathbb{R}$ uniquely determined by

$$\begin{aligned}\nu(J) &= 1, \\ \nu(x) &= 0 \text{ for each other join-irreducible } x \in \mathcal{G}_n\end{aligned}$$

is then invariant under the automorphisms of \mathcal{G}_n . However, ν cannot lie in \mathcal{C}_n . Indeed, by the very definition of χ_k , it follows at once that any element of \mathcal{C}_n assigns the same value to join-irreducible elements of the same height, because each χ_k has the latter property. This shows that $\mathcal{C}_n \neq \mathcal{I}_n$. It remains to show that $\mathcal{C}_n \subseteq \mathcal{I}_n$. This holds because each automorphism of \mathcal{G}_n carries a join-irreducible of a given height to a join-irreducible of the same height. \square

Finally, we turn to the announced result on the absence of linear relations among the χ_k 's.

Theorem 3.3. *For each integer $n \geq 1$, the set $\{\chi_1, \dots, \chi_{n+1}\}$ is a basis of \mathcal{C}_n . In particular, $\dim \mathcal{C}_n = n + 1$.*

Proof. Let again \mathcal{F}_n be the forest of join-irreducible elements of \mathcal{G}_n . As remarked at the beginning of this section, the height of \mathcal{F}_n – i.e. the cardinality of the longest chain in \mathcal{F}_n – is $n + 1$. Let us display such a chain

$$c_1 < c_2 < \dots < c_{n+1} .$$

Suppose that there are real numbers $r_1, \dots, r_{n+1} \in \mathbb{R}$ such that

$$r_1\chi_1 + \dots + r_{n+1}\chi_{n+1} = 0 , \quad (\star)$$

with the intention of showing $r_1 = \dots = r_{n+1} = 0$. By Definition 2.1, the evaluation of (\star) at c_i , for each $1 \leq i \leq n + 1$, yields the system of equations

$$\left\{ \begin{array}{l} r_1 + r_2 + \dots + r_i + r_{i+1} + \dots + r_{n+1} = 0 \\ \vdots \\ r_1 + 2r_2 + \dots + ir_i + ir_{i+1} + \dots + ir_{n+1} = 0 \\ \vdots \\ r_1 + 2r_2 + \dots + ir_i + (i+1)r_{i+1} + \dots + (n+1)r_{n+1} = 0 \end{array} \right. \quad (\text{S})$$

The determinant of the system (S) is

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 \\ 1 & 2 & 3 & \dots & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & n & n \\ 1 & 2 & 3 & \dots & n & n+1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} = 1 .$$

It follows that the system (S) has a unique solution, namely, $r_1 = \dots = r_{n+1} = 0$. \square

Remark. The generalised characteristics are integer-valued: their range is contained in the set of integers $\mathbb{Z} \subseteq \mathbb{R}$. Linear combinations with integer coefficients of generalised characteristics are again integer-valued. Therefore, if we write $\mathcal{C}_n^{\mathbb{Z}} \subseteq \mathcal{C}_n$ for the set of such linear combinations of generalised

characteristics of \mathcal{G}_n , then $\mathcal{C}_n^{\mathbb{Z}}$ has the structure of a \mathbb{Z} -module. In the proof above of Theorem 3.3, the fact that the determinant of the system (S) has value 1 – i.e. that the matrix of coefficients of (S) is *unimodular* – can be used to prove that $\mathcal{C}_n^{\mathbb{Z}}$ contains *all* integer-valued valuations of \mathcal{C}_n .

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