# The Euler Characteristic of a Formula in Gödel Logic

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***Abstract*—Using the lattice-theoretic version of the Euler characteristic introduced by V. Klee and G.-C. Rota, we define the Euler characteristic of a formula in Gödel logic (over finitely or infinitely many truth-values). We then prove that the information encoded by the Euler characteristic is classical, i.e., coincides with the analogous notion defined over Boolean logic. Building on this, we define $k$-valued versions of the Euler characteristic of a formula $\varphi$, for each integer $k \geq 2$, and prove that they indeed provide information about the logical status of $\varphi$ in Gödel $k$-valued logic. Specifically, our main result shows that the $k$-valued Euler characteristic is an invariant that separates $k$-valued tautologies from non-tautologies.**

## I. Introduction, Background, and Statement of Main Results

Some decades ago, V. Klee and G.-C. Rota introduced a lattice-theoretic analogue of the Euler characteristic, the celebrated topological invariant of polyhedra. Let us recall their definition. Let $L$ be a distributive lattice. A function $\nu\colon L \to \mathbb{R}$ is a *valuation* if it satisfies

$$\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y) \qquad (1)$$

for all $x, y, z \in L$. Recall that an element $x \in L$ is *join irreducible* if it is not the bottom element of $L$, and $x = y \vee z$ implies $x = y$ or $x = z$ for all $y, z \in L$. When $L$ is finite, it turns out [1, Corollary 2] that any valuation $\nu$ is uniquely determined by its values at the join-irreducible elements of $L$, along with its value at the bottom element $\bot$ of $L$.

*Definition 1.1 ([2, p. 120], [1, p. 36]):* The Euler characteristic of a finite distributive lattice $L$ is the unique valuation $\chi\colon L \to \mathbb{R}$ such that $\chi(x) = 1$ for any join-irreducible element $x \in L$, and $\chi(\bot) = 0$.

*Gödel* (*infinite-valued propositional*) *logic* $\mathbb{G}_\infty$ can be syntactically defined as the schematic extension of the intuitionistic propositional calculus by the *prelinearity axiom* $(\alpha \to \beta) \vee (\beta \to \alpha)$. It can also be semantically defined as a many-valued logic, as follows. Write FORM for the set of formulæ over propositional variables $X_1, X_2, \ldots$ in the language $\wedge, \vee, \to, \neg, \bot, \top$. (Here, $\bot$ and $\top$ are the logical constants *falsum* and *verum*, respectively.) An *assignment* is a function $\mu\colon \mathrm{FORM} \to [0,1] \subseteq \mathbb{R}$ with values in the real unit interval such that, for any two $\alpha,\ \beta \in \mathrm{FORM}$,

$$\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$$
$$\mu(\alpha \vee \beta) = \max\{\mu(\alpha), \mu(\beta)\}$$
$$\mu(\alpha \to \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases}$$

and $\mu(\neg\alpha) = \mu(\alpha \to \bot)$, $\mu(\bot) = 0$, $\mu(\top) = 1$. A *tautology* is a formula $\alpha$ such that $\mu(\alpha) = 1$ for every assignment $\mu$. As is well known, Gödel logic is complete with respect to this many-valued semantics. Indeed, for $\alpha \in \mathrm{FORM}$, let us write $\vdash \alpha$ to mean that $\alpha$ is derivable from the axioms of $\mathbb{G}_\infty$ using *modus ponens* as the only deduction rule. Then the completeness theorem guarantees that $\vdash \alpha$ holds if and only if $\alpha$ is a tautology. A stronger result holds: like classical logic, $\mathbb{G}_\infty$ also enjoys completeness for theories. For proofs and background, see [3].

For an integer $n \geq 1$, let us write $\mathrm{FORM}_n$ for the set of all formulæ whose propositional variables are contained in $\{X_1, \ldots, X_n\}$. As usual, $\varphi, \psi \in \mathrm{FORM}_n$ are called *logically equivalent* if both $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$ hold. Logical equivalence is an equivalence relation, written $\equiv$, and its equivalence classes are denoted $[\varphi]_\equiv$. By a routine check, the quotient set $\mathrm{FORM}_n/\equiv$ endowed with operations $\wedge$, $\vee$, $\top$, $\bot$ induced from the corresponding logical connectives becomes a distributive lattice with top and bottom element $\top$ and $\bot$, respectively. When $\mathrm{FORM}_n/\equiv$ is further endowed with the operation $\to$ induced by implication, it becomes a Heyting algebra satisfying prelinearity; such algebras we call *Gödel algebras* (cf. the term *G-algebras* in [3, 4.2.12]). The specific Gödel algebra $\mathscr{G}_n = \mathrm{FORM}_n/\equiv$ is, by construction, the *Lindenbaum algebra* of Gödel logic over the language $\{X_1, \ldots, X_n\}$.

It is a remarkable fact due to Horn [4, Theorem 4] that $\mathscr{G}_n$ is finite for each integer $n \geq 1$, in analogy with Boolean algebras. A second important fact is that a finite Heyting algebra $H$ is a Gödel algebra if and only if its collection of join-irreducible elements, ordered by restriction from $H$, is a *forest*; i.e., the lower bounds of any such element are a totally ordered set. A more general version of this result is also due to Horn [5, Theorem 2.4].

Knowing that $\mathscr{G}_n$ is a finite distributive lattice whose elements are formulæ in $n$ variables, up to logical equivalence, one is led to give the following definition.

*Definition 1.2:* The *Euler characteristic* of a formula $\varphi \in \mathrm{FORM}_n$, written

$$\chi(\varphi)\ ,$$

is the number $\chi([\varphi]_\equiv)$, where $\chi$ is the Euler characteristic of the finite distributive lattice $\mathscr{G}_n$.

However, the question is now whether $\chi(\varphi)$ encodes genuinely *logical* information about $\varphi$, just like the Euler characteristic of a polyhedron provides geometric information about that polyhedron. The answer turns out to be affirmative. As usual, we say an assignment $\mu\colon \text{FORM}_n \to [0,1]$ is *Boolean* if it takes values in $\{0,1\}$.

*Theorem 1.3:* Fix an integer $n \geq 1$. For any formula $\varphi \in \text{FORM}_n$, the Euler characteristic $\chi(\varphi)$ equals the number of Boolean assignments $\mu\colon \text{FORM}_n \to [0,1]$ such that $\mu(\varphi) = 1$.

As an immediate consequence of Theorem 1.3,

$$0 \leq \chi(\varphi) \leq 2^n$$

for any $\varphi \in \text{FORM}_n$. In particular, note that the following hold.

- If $\varphi$ is a tautology in $\mathbb{G}_\infty$, then $\chi(\varphi) = 2^n$.
- If $\chi(\varphi) = 2^n$, then $\varphi$ is a tautology in classical propositional logic.
- If $\chi(\varphi) = 0$, then $\varphi$ is a contradiction in classical propositional logic, and conversely.

Theorem 1.3 shows that, while $\chi(\varphi)$ does encode non-trivial logical information, that information is classical, and independent of Gödel logic. In fact, if one replicates the above construction over classical logic, one ends up with a valuation $\chi$ on the Boolean algebra of $n$-variable formulæ that simply counts the number of atoms below each element in the Boolean algebra. By the same token, the Euler characteristic cannot tell apart the tautologies in Gödel logic from the remaining formulæ, whereas it does so for classical tautologies. We now show how to remedy this by considering a different valuation on $\mathscr{G}_n$. The *height* of a join irreducible $g \in \mathscr{G}_n$ is the length $l$ of the longest chain $g = g_1 > g_2 > \cdots > g_l$ in $\mathscr{G}_n$ with each $g_i$ a join-irreducible element. We write $h(g)$ for the height of $g$.

*Definition 1.4:* Fix integers $n, k \geq 1$. We write $\chi_k\colon \mathscr{G}_n \to \mathbb{R}$ for the unique valuation on $\mathscr{G}_n$ that satisfies

$$\chi_k(g) = \min\{h(g), k\}$$

for each join-irreducible element $g \in \mathscr{G}_n$, and such that, moreover, $\chi_k(\bot) = 0$. Further, if $\varphi \in \text{FORM}_n$, we define $\chi_k(\varphi) = \chi_k([\varphi]_\equiv)$.

Clearly, $\chi_1$ is the Euler characteristic $\chi$. Our main result shows that $\chi_k$ is a natural generalisation of $\chi$ in that it tells apart the tautologies in *Gödel* $(k+1)$*-valued logic* $\mathbb{G}_{k+1}$ from the remaining formulæ. We recall that $\mathbb{G}_{k+1}$ is the schematic extension of $\mathbb{G}_\infty$ via

$$\alpha_1 \vee (\alpha_1 \to \alpha_2) \vee \cdots \vee (\alpha_1 \wedge \cdots \wedge \alpha_k \to \alpha_{k+1})\,. \qquad (2)$$

Semantically, restrict assignments to those taking values in the set

$$V_{k+1} = \{0 = \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \frac{k}{k} = 1\} \subseteq [0,1]\ ,$$

that is, to $(k+1)$*-valued assignments*. A tautology of $\mathbb{G}_{k+1}$ is defined as a formula that takes value $1$ under any such assignment. Then $\mathbb{G}_{k+1}$ is complete with respect to this semantics; see e.g. [6] for further background. To state our main result, we need one more definition (cf. [7, Definition 2.1]).

*Definition 1.5:* Fix integers $n, k \geq 1$. We say that two $(k+1)$-valued assignments $\mu$ and $\nu$ are *equivalent over the first* $n$ *variables*, or just $n$*-equivalent*, written $\mu \equiv_n^k \nu$, if and only if there exists a permutation $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ such that

$$\begin{aligned} 0 \preceq_0 \mu(X_{\sigma(1)}) \preceq_1 \cdots \preceq_{n-1} \mu(X_{\sigma(n)}) \preceq_n 1\ ,\\ 0 \preceq_0 \nu(X_{\sigma(1)}) \preceq_1 \cdots \preceq_{n-1} \nu(X_{\sigma(n)}) \preceq_n 1\ , \end{aligned} \qquad (3)$$

where $\preceq_i \in \{<,=\}$, for $i = 0,\ldots,n$.

Thus, two assignments are equivalent if and only if they are indiscernible on the grounds of the strict inequality ($<$) and equality ($=$) relations that hold among the values that each of them assigns to propositional variables. Clearly, $\equiv_n^k$ is an equivalence relation. In various guises, the notion above of equivalent assignments plays a crucial rôle in the investigation of Gödel logic; see e.g. [7], [8]. For our purposes here, we observe that distinct 2-valued (=Boolean) assignments are not equivalent, so that there are $2^n$ equivalence classes of such assignments over the first $n$ variables.

We next introduce the $(k+1)$-valued Gödelian analogue of the number $2^n$. As will be proved in Subsection II-A, the following recursive formula counts the number of join-irreducible elements of $\mathscr{G}_n$ having height smaller or equal than $k$.

$$P(n,k) = \sum_{i=1}^{k} \sum_{j=0}^{n} \binom{n}{j} T(j,i)\,, \qquad (*)$$

where

$$T(n,k) = \begin{cases} 1 & \text{if } k = 1\,,\\ 0 & \text{if } k > n+1\,,\\ \displaystyle\sum_{i=1}^{n} \binom{n}{i} T(n-i,k-1) & \text{otherwise}\,. \end{cases}$$

| | k=1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| n=1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 |
| 2 | 4 | 9 | 11 | 11 | 11 | 11 | 11 |
| 3 | 8 | 27 | 45 | 51 | 51 | 51 | 51 |
| 4 | 16 | 81 | 191 | 275 | 299 | 299 | 299 |
| 5 | 32 | 243 | 813 | 1563 | 2043 | 2163 | 2163 |
| 6 | 64 | 729 | 3431 | 8891 | 14771 | 18011 | 18731 |
| 7 | 128 | 2187 | 14325 | 49731 | 106851 | 158931 | 184131 |
| 8 | 256 | 6561 | 59231 | 272675 | 757019 | 1407179 | 1921259 |
| 9 | 512 | 19683 | 242973 | 1468203 | 5228043 | 12200883 | 20214483 |

TABLE I

THE NUMBER $P(n,k)$ OF DISTINCT EQUIVALENCE CLASSES OF $(k+1)$-VALUED ASSIGNMENTS OVER $n$ VARIABLES.

*Theorem 1.6:* Fix integers $n, k \geq 1$, and a formula $\varphi \in \text{FORM}_n$.

1) $\chi_k(\varphi)$ equals the number of $(k+1)$-valued assignments $\mu\colon \text{FORM}_n \to [0,1]$ such that $\mu(\varphi) = 1$, up to $n$-equivalence.
2) $\varphi$ is a tautology in $\mathbb{G}_{k+1}$ if and only if $\chi_k(\varphi) = P(n,k)$.
3) $\varphi$ is a tautology in $\mathbb{G}_\infty$ if and only if it is a tautology in $\mathbb{G}_{n+2}$ if and only if $\chi_{n+1}(\varphi) = P(n, n+1)$.

Since it is readily verified that distinct 2-valued (=Boolean) assignments are not equivalent, Theorem 1.3 is an immediate consequence of Theorem 1.6. We note that

$$P(n,1) = \sum_{j=0}^{n} \binom{n}{j} T(j,1) = \sum_{j=0}^{n} \binom{n}{j} = 2^n \,,$$

a circumstance supporting our previous statement that (*) generalises $2^n$ to $(k+1)$-valued Gödel logic. We currently do not know whether there exists a more concise closed formula for the number $P(n,k)$.

We now turn to the proof of our main result.

## II. Proof of Theorem

### A. Proof of (*)

Let $\mathscr{F}_n$ be the forest of join-irreducible elements of $\mathscr{G}_n$, and let $\mathscr{T}_n$ by the unique tree of $\mathscr{F}_n$ having maximum height (cf. [8, Section 2.3]). By the height $h(F)$ of a forest $F$ we mean the cardinality of its longest chain. Denote by $\uparrow g$ the upper set of an element $g$. Recall that an *atom* of a partially ordered set with minimum is an element that covers its minimum. It can be shown (cf. [8, Lemma 2.3 – (a)]) that $\mathscr{T}_n$ has precisely $\binom{n}{i}$ atoms $a$ with $\uparrow a \cong \mathscr{T}_{n-i}$, for each $i = 1, \ldots, n$, and no other atom. Observing that $\mathscr{T}_0$ is the one-element tree, and that $h(\mathscr{T}_n) = h(\mathscr{T}_{n-1}) + 1$ for each $n$, we immediately obtain the following recursive formula for the number of elements of $\mathscr{T}_n$ having height $k$.

$$T(n,k) = \begin{cases} 1 & \text{if } k = 1\,, \\ 0 & \text{if } k > n+1\,, \\ \displaystyle\sum_{i=1}^{n} \binom{n}{i} T(n-i, k-1) & \text{otherwise}\,. \end{cases}$$

Further, $\mathscr{F}_n$ contains precisely $\binom{n}{i}$ distinct copies of $\mathscr{T}_i$, for $i = 0, \ldots, n$, and no other tree (cf. [8, Lemma 2.3 – (b)].) Thus, as claimed, $P(n,k)$ gives the number of elements of $\mathscr{F}_n$ having height smaller or equal than $k$ (i.e., the number of join-irreducible elements of $\mathscr{G}_n$ having height smaller or equal than $k$.)

### B. Two lemmas

*Lemma 2.1:* Fix integers $n, k \geq 1$, let $x \in \mathscr{G}_n$ and consider the valuation $\chi_k\colon \mathscr{G}_n \to \mathbb{R}$. Then, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq x$ and $h(g) \leq k$.

*Proof:* If $x = \bot$ then, by Definition 1.4, $\chi_k(x) = 0$, and the Lemma trivially holds. Otherwise, we argue as follows.

Let $F$ be the forest of all join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq x$. (Recall that $x$ is the join of the join-irreducible elements $g \in F$.) We proceed by induction on the structure of $F$. If $F$ is the one-element forest, then $x$ is a join-irreducible element, and $F = \{x\}$. By Definition 1.4, $\chi_k(x) = 1$, for each $k \geq 1$, as desired.

Let now $|F| > 1$. Let $l \in F$ be a maximal element of $F$, and consider the forest $F^- = F \setminus \{l\}$. Let $x^-$ be the join of the elements of $F^-$. We immediately observe that $x = l \vee x^-$.

If $l$ is an atom of $\mathscr{G}_n$, then $l \wedge x^- = \bot$. By (1) and Definition 1.4, $\chi_k(x) = \chi_k(l \vee x^-) = \chi_k(l) + \chi_k(x^-) - \chi_k(l \wedge x^-) = 1 + \chi_k(x^-)$. Using the inductive hypotheses on $F^-$ we obtain our statement, for the case $h(l) = 1$.

Let, finally, $h(l) > 1$. Consider the element $l^- = l \wedge x^-$. Let $L$ be the forest of all join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq l$, and let $L^-$ be the forest of all join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq l^-$. Since $l$ is a join irreducible, $L$ is a chain. Moreover, one easily see that $L^- = L \setminus \{l\}$. For a forest $P$, we denote by $|P|_k$ the number of elements $p$ of $P$ such that $h(p) \leq k$. We consider two cases.

$h(l) \leq k$. We observe that $|F^-|_k = |F|_k - 1$ and that $|L^-|_k = |L|_k - 1$. Using (1) and the inductive hypotheses we obtain $\chi_k(x) = \chi_k(l) + \chi_k(x^-) - \chi_k(l \wedge x^-) = |L|_k + |F|_k - 1 - (|L|_k - 1) = |F|_k$. In other words, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq x$ and $h(g) \leq k$.

$h(l) > k$. In this case, we observe that $|F^-|_k = |F|_k$ and that $|L^-|_k = |L|_k$. Using (1) and the inductive hypotheses we obtain $\chi_k(x) = \chi_k(l) + \chi_k(x^-) - \chi_k(l \wedge x^-) = |L|_k + |F|_k - |L|_k = |F|_k$. In other words, $\chi_k(x)$ equals the number of join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq x$ and $h(g) \leq k$, and the lemma is proved. ■

*Lemma 2.2:* Fix integers $n, k \geq 1$, and let $\varphi \in \text{FORM}_n$. Let $O(\varphi, n, k)$ be the set of equivalence classes $[\mu]_{\equiv_n^k}$ of $(k+1)$-valued assignments such that $\mu(\varphi) = 1$. Further, let $J(\varphi, n, k)$ be the set of join-irreducible elements $x \in \mathscr{G}_n$ such that $x \leq [\varphi]_\equiv$ and $h(x) \leq k$. Then there is a bijection between $O(\varphi, n, k)$ and $J(\varphi, n, k)$.

*Proof:* In the proof of this lemma we use techniques from algebraic logic; for all unexplained notions, please see [3].

Fix a $(k+1)$-valued assignments $\mu\colon \text{FORM}_n \to V_{k+1}$. Endow $V_{k+1}$ with its unique structure of Gödel algebra compatible with the total order of the elements of $V_{k+1} \subseteq [0,1]$. Then there is a unique homomorphism of Gödel algebras $h_\mu\colon \mathscr{G}_n \to V_{k+1}$ corresponding to $\mu$, namely,

$$h_\mu([\varphi]_\equiv) = \mu(\varphi)\ . \tag{4}$$

Conversely, given any such homomorphism $h\colon \mathscr{G}_n \to V_{k+1}$, there is a unique $(k+1)$-valued assignment $\mu_h\colon \text{FORM}_n \to V_{k+1}$ corresponding to $h$, namely,

$$\mu_h(\varphi) = h([\varphi]_\equiv)\ . \tag{5}$$

Clearly, the correspondences (4–5) are mutually inverse, and thus yield a bijection between $(k+1)$-valued assignments to FORM and homomorphisms $\mathscr{G}_n \to V_{k+1}$. Further, upon noting that $\mu_h(\varphi) = 1$ in (5) if and only if $h_\mu([\varphi]) = 1$ in (4), we see that this bijection restricts to a bijection

$$O'(\varphi, n, k) \cong \hom(\varphi, \mathscr{G}_n, V_{k+1}) \tag{6}$$

where the right-hand side is the set of homomorphisms $h\colon \mathscr{G}_n \to V_{k+1}$ such that $h([\varphi]_\equiv) = 1$, while the left-hand side is the collection of $(k+1)$-valued assignments $\mu\colon \text{FORM}_n \to V_{k+1}$ with $\mu(\varphi) = 1$. Now recall that to any homomorphism $h\colon \mathscr{G}_n \to V_{k+1}$ one associates the prime (lattice) filter of $\mathscr{G}_n$ given by $\mathfrak{p}_h = h^{-1}(1)$. Conversely, given a prime filter $\mathfrak{p}$ of $\mathscr{G}_n$ there is a natural onto quotient map $h_\mathfrak{p}\colon \mathscr{G}_n \twoheadrightarrow \mathscr{G}_n/\mathfrak{p}$, where $C = \mathscr{G}_n/\mathfrak{p}$ is a chain of finite cardinality; further, $|C| \leq k+1$ if and only if $\mathfrak{p}$ has height $\leq k$, meaning that the chain of prime filters containing it has cardinality $k$. Since any chain with $|C| \leq k+1$ embeds into $V_{k+1}$, this shows that each prime filter $\mathfrak{p}$ of $\mathscr{G}_n$ having height $\leq k$ induces by

$$h^e_\mathfrak{p}\colon \mathscr{G}_n \twoheadrightarrow \mathscr{G}_n/\mathfrak{p} \overset{e}{\hookrightarrow} V_{k+1} \tag{7}$$

one homomorphism (not necessarily onto) $h^e_\mathfrak{p}$ from $\mathscr{G}_n$ to $V_{k+1}$ for *each* choice of the embedding $e\colon \mathscr{G}_n/\mathfrak{p} \hookrightarrow V_{k+1}$. It is now easy to check that two $(k+1)$-valued assignments $\mu, \nu\colon \text{FORM}_n \to V_{k+1}$ satisfy $\mu \equiv^k_n \nu$ if and only if the associated homomorphisms $h_\mu, h_\nu$ as in (4) factor as in (7) for the *same* prime filter $\mathfrak{p}$, although for possibily different embeddings $e$ and $e'$ into $V_{k+1}$. It is clear that this yields an equivalence relation on such homomorphisms $h_\mu, h_\nu$. Let us denote by $\hom_\equiv(\varphi, \mathscr{G}_n, V_{k+1})$ the set of equivalence classes of those homomorphisms $h_\mu$ satisfying $h_\mu([\varphi]_\equiv) = 1$. Summing up, from the bijection in (6) we obtain a bijection

$$O(\varphi, n, k) \cong \hom_\equiv(\varphi, \mathscr{G}_n, V_{k+1}) \ . \tag{8}$$

To complete the proof, observe that since $\mathscr{G}_n$ is finite, every filter $\mathfrak{p}$ of $\mathscr{G}_n$ is principal, i.e., if there is an element $p \in \mathscr{G}_n$ such that $\mathfrak{p} = \uparrow p$; moreover, $\mathfrak{p}$ is prime if and only if $p$ is join irreducible. In other words, there is a bijection between join-irreducible elements and prime filters of $\mathscr{G}_n$. By definition, the natural quotient map $\mathscr{G}_n \twoheadrightarrow \mathscr{G}_n/\mathfrak{p}$ sends $[\varphi]_\equiv$ to 1 if and only if $[\varphi]_\equiv$ lies in the prime filter $\mathfrak{p}$; that is, if and only if $[\varphi]_\equiv \geq p$ in $\mathscr{G}_n$. Moreover, the following is easily checked. Suppose $\mathfrak{p} = \uparrow p$ as in the above, and let $\mathscr{G}_n/\uparrow p$ be the quotient algebra, which is a chain because $\mathfrak{p}$ is prime. Then $|\mathscr{G}_n/\uparrow p| \leq k+1$ if and only if the height of $p$ satisfies $h(p) \leq k$. Using the preceding observations, from (7) and the definition of $\hom_\equiv(\varphi, \mathscr{G}_n, V_{k+1})$ we obtain a bijection

$$\hom_\equiv(\varphi, \mathscr{G}_n, V_{k+1}) \cong J(\varphi, n, k) \ . \tag{9}$$

The lemma follows from (8) and (9). ∎

### C. *End of Proof of Theorem 1.6*

1) By Lemma 2.1 the value $\chi_k(\varphi) = \chi_k([\varphi]_\equiv)$ is given by the number of join-irreducible elements $g \in \mathscr{G}_n$ such that $g \leq [\varphi]_\equiv$ and $h(g) \leq k$. By Lemma 2.2, such number equals the number of equivalence classes $[\mu]_{\equiv^k_n}$ of $(k+1)$-valued assignments such that $\mu(\varphi) = 1$, and the statement follows.
2) As proved in Subsection II-A, the formula $P(n,k)$ counts the total number of join-irreducible elements of $\mathscr{G}_n$ having height smaller or equal than $k$. By Lemma 2.1, $\chi_k(\varphi) = P(n,k)$ if and only if all the join-irreducible elements $g \in \mathscr{G}_n$ such that $h(g) \leq k$ satisfy $g \leq [\varphi]_\equiv$. By Lemma 2.2, the latter holds if and only if each $(k+1)$-valued assignment $\mu\colon \text{FORM}_n \to [0,1]$ satisfies $\mu(\varphi) = 1$, i.e., $\varphi$ is a tautology in $\mathbb{G}_{k+1}$, as desired.
3) *Claim:* If $\varphi \in \text{FORM}_n$ is a tautology in $\mathbb{G}_{n+2}$, then it is a tautology in $\mathbb{G}_\infty$.

   *Proof of Claim:* Suppose, by way of contradiction, that $\varphi$ is not a tautology in $\mathbb{G}_\infty$, but it is a tautology in $\mathbb{G}_{n+2}$. Thus, there must exists an assignment $\mu$ such that $\mu(\varphi) < 1$. An easy structural induction shows that $\mu(\varphi) \in \{0, \mu(X_1), \ldots, \mu(X_n), 1\}$. But then, the restriction of $\mu$ onto its image yields an $(n+2)$-valued assignment $\bar{\mu}$ such that $\bar{\mu}(\varphi) < 1$, a contradiction.

   As one can immediately check, if $\varphi$ is a tautology in $\mathbb{G}_\infty$, then it is a tautology in $\mathbb{G}_{n+2}$. Thus, using the Claim, $\varphi$ is a tautology in $\mathbb{G}_{n+2}$ if and only if it is a tautology in $\mathbb{G}_\infty$. Finally, by the statement 2) of this theorem , $\varphi$ is a tautology in $\mathbb{G}_{n+2}$ if and only if $\chi_{n+1}(\varphi) = P(n, n+1)$, and the last statement of the theorem is proved.

### D. *An Example*

Let us consider the Gödel algebra $\mathscr{G}_1$, depicted in Figure 1.

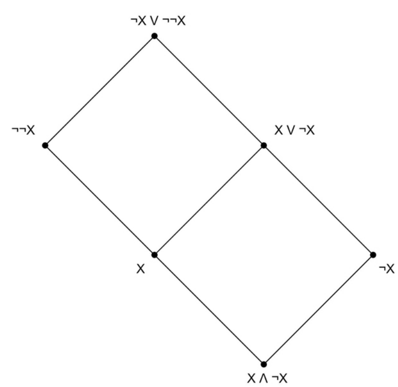


Fig. 1. The Gödel algebra $\mathscr{G}_1$.

Lemma 2.1 allows us to compute the values of $\chi_k(x)$ for each $x \in \mathscr{G}_1$, simply by counting the number of join-irreducible elements under $x$ having height not greater than $k$. The results are displayed in Figure 2, for $k = 1$ (i.e., for the Euler characteristic), and for $k = 2$. Note that for $k \geq 3$ and for each $x \in \mathscr{G}_1$, $\chi_k(x)$ and $\chi_2(x)$ coincide, by the third statement of Theorem 1.6.

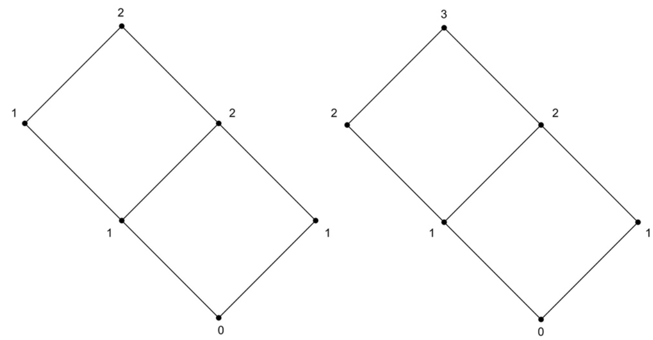


Fig. 2. The values of $\chi_1 \colon \mathscr{G}_1 \to \mathbb{R}$ (to the left) and $\chi_2 \colon \mathscr{G}_1 \to \mathbb{R}$ (to the right).

Let us consider the formula $\neg\neg X$. One can check that, up to $n$-equivalence, there are two distinct 3-valued assignments $\mu, \nu \colon \text{FORM}_1 \to \{0, \frac{1}{2}, 1\}$ such that $\mu(\neg\neg X) = \nu(\neg\neg X) = 1$. Namely, we can take $\mu$ such that $\mu(X) = 1$, and $\nu$ such that $\nu(X) = \frac{1}{2}$. In fact, as one sees in Figure 2, $\chi_2(\neg\neg X) = 2$. The assignment $\mu(X)$ is the only Boolean assignment such that $\mu(\neg\neg X) = 1$. Actually, $\chi_1(\neg\neg X) = \chi(\neg\neg X) = 1$.

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