

An Analysis of Ruspini Partitions in Gödel Logic

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Abstract

By a *Ruspini partition* we mean a finite family of fuzzy sets $\{f_1, \dots, f_n\}$, $f_i : [0, 1] \rightarrow [0, 1]$, such that $\sum_{i=1}^n f_i(x) = 1$ for all $x \in [0, 1]$, where $[0, 1]$ denotes the real unit interval. We analyze such partitions in the language of Gödel logic. Our first main result identifies the precise degree to which the Ruspini condition is expressible in this language, and yields *inter alia* a constructive procedure to axiomatize a given Ruspini partition by a theory in Gödel logic. Our second main result extends this analysis to Ruspini partitions fulfilling the natural additional condition that each f_i has at most one left and one right neighbour, meaning that $\min_{x \in [0, 1]} \{f_{i_1}(x), f_{i_2}(x), f_{i_3}(x)\} = 0$ holds for $i_1 \neq i_2 \neq i_3$.

Key words: Fuzzy set, Ruspini partition, Gödel logic.

1 Introduction

Let $[0, 1]$ be the real unit interval. By a *fuzzy set* we shall mean a function $f : [0, 1] \rightarrow [0, 1]$. Throughout the paper, we fix a finite nonempty family

$$P = \{f_1, \dots, f_n\}$$

of fuzzy sets, for $n \geq 1$ an integer. Moreover, we write \underline{n} for the set $\{1, \dots, n\}$.

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In several soft computing applications, the following notion of fuzzy partition plays an important role. It is often traced back to [1, p. 28].

Definition 1.1 We say P is a Ruspini partition if for all $x \in [0, 1]$

$$\sum_{i=1}^n f_i(x) = 1. \quad (1.1)$$

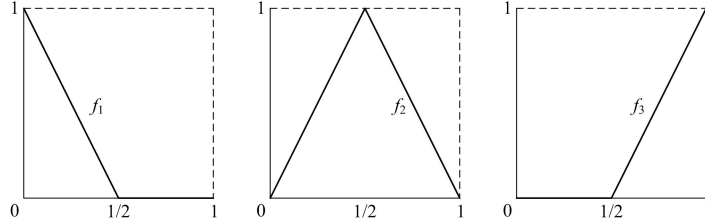


Fig. 1. A Ruspini partition $\{f_1, f_2, f_3\}$.

By way of informal motivation for what follows, think of the real unit interval $[0, 1]$ as the normalized range of values of a physical observable, say temperature. Then each $f_i \in P$ can be viewed as a means of assigning a truth-value to a proposition about temperature in some many-valued logic \mathcal{L} . Had one no information at all about such propositions, one would be led to identify them with propositional variables X_i , subject only to the axioms of \mathcal{L} . However, the set P does encode information about X_1, \dots, X_n . For example, consider $P = \{f_1, f_2, f_3\}$ as in Fig. 1, and say f_1 , f_2 , and f_3 provide truth-values for the propositions $X_1 =$ “The temperature is low”, $X_2 =$ “The temperature is medium”, and $X_3 =$ “The temperature is high”, respectively. If \mathcal{L} has a conjunction \wedge interpreted by minimum, the proposition $X_1 \wedge X_3$ has 0 as its only possible truth-value, *i.e.*, it is a contradiction. The chosen set P then leads one to add extra-logical axioms to \mathcal{L} , *e.g.*, $\neg(X_1 \wedge X_3)$, in an attempt to express the fact that one cannot observe both a high and a low temperature at the same time. More generally, P implicitly encodes a *theory*—that is, a family of formulas required to hold, thought of as extra-logical axioms—over the pure logic \mathcal{L} . Imposing the Ruspini condition on P , then, amounts to implicitly enriching the logic \mathcal{L} by extra-logical axioms that attempt to capture condition (1.1) in the language provided by \mathcal{L} . Indeed, while in practice it is often the case that \mathcal{L} lacks the power to express addition of real numbers exactly, \mathcal{L} will still afford an approximation of the Ruspini condition *in its own language*. In this paper we are thus concerned with the general problem of making explicit the extra-logical information implicitly encoded by P .

Throughout this paper, we shall take \mathcal{L} to be Gödel logic. Among triangular norms and conorms [2], the minimum and maximum operators are rather popular choices to model fuzzy logical conjunction and disjunction in applications. Gödel logic adds to this setting an implication that is obtained from conjunction via *residuation*, and thus fits into P. Hájek’s family of fuzzy logics based

on (continuous) triangular norms; we refer to [3] for an extensive treatment.

Here we recall that *Gödel (infinite-valued propositional) logic* \mathbb{G}_∞ can be syntactically defined as the schematic extension of the intuitionistic propositional calculus by the *prelinearity axiom* $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$. It can also be semantically defined as a many-valued logic, as follows. Let us consider well-formed formulas over propositional variables X_1, X_2, \dots in the language $\wedge, \vee, \rightarrow, \neg, \perp, \top$. (We use \perp and \top as the logical constants *falsum* and *verum*, respectively). By an *assignment* we shall mean a function μ from (well-formed) formulas to $[0, 1] \subseteq \mathbb{R}$ such that, for any two such formulas α, β ,

$$\begin{aligned} \mu(\alpha \wedge \beta) &= \min\{\mu(\alpha), \mu(\beta)\} \\ \mu(\alpha \vee \beta) &= \max\{\mu(\alpha), \mu(\beta)\} \\ \mu(\alpha \rightarrow \beta) &= \begin{cases} 1 & \text{if } \mu(\alpha) \leq \mu(\beta) \\ \mu(\beta) & \text{otherwise} \end{cases} \end{aligned}$$

and $\mu(\neg\alpha) = \mu(\alpha \rightarrow \perp)$, $\mu(\perp) = 0$, $\mu(\top) = 1$. A *tautology* is a formula α such that $\mu(\alpha) = 1$ for every assignment μ . As is well known, Gödel logic is complete with respect to this many-valued semantics. Indeed, for α a formula of \mathbb{G}_∞ , let us write $\vdash \alpha$ to mean that α is derivable from the axioms of \mathbb{G}_∞ using *modus ponens* as the only deduction rule. Then the completeness theorem guarantees that $\vdash \alpha$ holds if and only if α is a tautology. For proofs and more details, see [3], [4].

This paper provides a thorough analysis of how the Ruspini condition on P is reflected by its associated theory over Gödel logic. In (3.6) we shall eventually obtain a constructive procedure to axiomatize the theory implicitly encoded by P . Gödel logic cannot precisely capture addition of real numbers, and Theorem 3.10 in fact proves that—up to logical equivalence in \mathbb{G}_∞ —the Ruspini condition (1.1) reduces to the notion of *weak Ruspini partition* given in Definition 3.4. In Section 2 we collect the necessary algebraic and combinatorial background, and prove some preliminary results. Theorem 3.10 is proved in Section 3.

In several applications, the family of fuzzy sets P satisfies additional requirements beyond the Ruspini condition. Indeed, designers often prefer fuzzy sets that have at most one neighbour to the left and one neighbour to the right, as in Figure 1. If, by contrast, one allows configurations such as the one in Figure 2, one contemplates the possibility that certain values of the physical observable—temperature, in our example—are at the same time low, medium, and high (to possibly different degrees). While this may be what is called for by specific situations, it turns out that in many applications the membership functions are chosen so as to avoid this. Cf. *e.g.*, the majority of the examples in [5]. Formally, we consider the following definition.

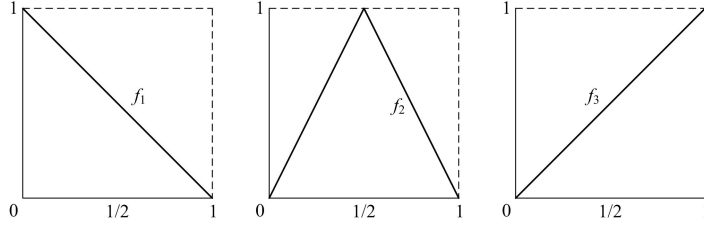


Fig. 2. A 3-overlapping family $\{f_1, f_2, f_3\}$.

Definition 1.2 We say P is 2-overlapping if for all $x \in [0, 1]$ and all triples of indices $i_1 \neq i_2 \neq i_3$ one has

$$\min \{f_{i_1}(x), f_{i_2}(x), f_{i_3}(x)\} = 0 . \quad (1.2)$$

The set P in Figure 1, for instance, is a 2-overlapping family. One could define k -overlapping families of fuzzy sets in the obvious manner. However, in this paper we shall only deal with the 2-overlapping case.

In Section 4, we subject a family P of 2-overlapping fuzzy sets to the same analysis carried out for the Ruspini condition. Indeed, Theorem 4.6 is the exact counterpart for condition (1.2) of Theorem 3.10. There are, however, two significant differences. Firstly, Gödel logic does capture the minimum of two real numbers exactly. This is why we do not need a weakened notion of 2-overlapping families of fuzzy sets, whereas for the Ruspini condition the concept of a weak Ruspini partition given in Definition 3.4 is unavoidable. Secondly, and more interestingly, the theory implicitly encoded by a family P of 2-overlapping fuzzy sets can already be axiomatized in *four-valued Gödel logic*, denoted \mathbb{G}_4 : even if n grows ever larger, it is not necessary to use more than four truth-values. The required background on finite-valued Gödel logics is recalled in Section 2.

This reduction to four-valued Gödel logic continues to hold when we assume that P satisfies both the Ruspini condition (1.1) and the 2-overlapping condition (1.2). Closing a circle of ideas, in our final Theorem 4.7 we obtain the axiomatic characterization (over \mathbb{G}_4) of those weak Ruspini partitions P that are 2-overlapping.

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2 Preliminary Results

In this section, our aim is twofold. First, we wish to associate with P a formula $\alpha_P(X_1, \dots, X_n)$ in Gödel logic that encodes all the extra-logical information provided by P , as discussed in the introduction. Second, we wish to explain how precisely the same information can be encoded in combinatorial terms using appropriate partially ordered sets (*posets*, for short). For this, we shall eventually associate with P (and α_P) a poset $F(P)$ (and F_{α_P})—see (2.7) below.

2.1 Gödel algebras

As a tool, we make use of the algebraic counterpart of Gödel logic, namely, *Gödel algebras*. These are Heyting algebras¹ $\langle G, \wedge, \vee, \rightarrow, \neg, \top, \perp \rangle$ satisfying the prelinearity condition $(x \rightarrow y) \vee (y \rightarrow x) = \top$. Thus, Gödel algebras are to Gödel logic precisely as Boolean algebras are to classical propositional logic. The standard correspondence between algebraic and logical notions generalizes to Gödel logic, and shall be used below.

The collection of all functions from $[0, 1]$ to $[0, 1]$ has the structure of a Gödel algebra under the following operations, for $f, g : [0, 1] \rightarrow [0, 1]$.

$$\begin{aligned} (f \wedge g)(x) &= \min \{f(x), g(x)\} & (f \vee g)(x) &= \max \{f(x), g(x)\} \\ (f \rightarrow g)(x) &= \begin{cases} 1 & \text{if } f(x) \leq g(x) \\ g(x) & \text{otherwise} \end{cases} & (\neg f)(x) &= \begin{cases} 1 & \text{if } f(x) = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The top and bottom elements of the algebra are the constant functions 1 and 0, respectively.

We shall denote by $\mathcal{G}(P)$ the Gödel subalgebra of the algebra of all functions from $[0, 1]$ to itself generated by P . For each integer $k \geq 0$, we write \mathcal{G}_k for the *free Gödel algebra* on k free generators x_1, \dots, x_k corresponding to the propositional variables X_1, \dots, X_k . That is, \mathcal{G}_k is the Lindenbaum algebra of the pure Gödel logic restricted to the first k propositional variables. Then \mathcal{G}_k is finite—it is well-known that Gödel algebras form a locally finite variety of algebras [8, Theorem 4]. Since $\mathcal{G}(P)$ is generated by the n elements f_1, \dots, f_n , there is a congruence Θ on \mathcal{G}_n such that the quotient algebra \mathcal{G}_n/Θ satisfies

$$\mathcal{G}_n/\Theta \cong \mathcal{G}(P), \quad (2.1)$$

¹ For background on Heyting algebras, we refer to [7].

where \cong denotes isomorphism of Gödel algebras. We recall that congruences of a Gödel algebra G are in one-one correspondence with *filters* of G , that is, with upward closed subsets closed under the \wedge operation. In particular, filters of the form $\uparrow x = \{y \in G \mid y \geq x\}$ are called *principal*, as their corresponding congruences. If, additionally, G is finite, all filters (and congruences) are necessarily principal. Therefore, Θ is generated by a single equation $\alpha(x_1, \dots, x_n) = \top$ in the language of Gödel algebras. In logical terms, there is a single formula

$$\alpha_P \equiv \alpha_P(X_1, \dots, X_n) \quad (2.2)$$

over the n propositional variables X_1, \dots, X_n , such that the Lindenbaum algebra of the theory axiomatized by the single axiom α_P is isomorphic to $\mathcal{G}(P)$. Note that α_P is uniquely determined by P up to logical equivalence. Indeed, if $\alpha(X_1, \dots, X_n)$ is another formula such that the corresponding equation $\alpha(x_1, \dots, x_n) = \top$ generates the congruence Θ , then, algebraically, α and α_P represent the unique element x of \mathcal{G}_n that generates the unique principal filter $\uparrow x$ corresponding to Θ . Hence, $\vdash \alpha \leftrightarrow \alpha_P$, where we write $\alpha \leftrightarrow \alpha_P$ as a shorthand for $(\alpha \rightarrow \alpha_P) \wedge (\alpha_P \rightarrow \alpha)$.

Intuitively, then, the formula α_P encodes *all* relations between the fuzzy sets f_1, \dots, f_n that Gödel logic is capable to express. The standard argument above only grants the existence and uniqueness of α_P , given P . We now turn to the problem of describing α_P concretely in terms of P .

2.2 Combinatorial representation

Any finite Boolean algebra can be thought of as the family of all subsets of a finite set, endowed with set-theoretic operations. For finite Gödel algebras, one needs to replace sets with *forests*, as follows.

Recall that, given a poset (F, \leq) and a set $Q \subseteq F$, the *downset* of Q is

$$\downarrow Q = \{x \in F \mid x \leq q, \text{ for some } q \in Q\}.$$

We write $\downarrow q$ for $\downarrow \{q\}$. A poset F is a *forest* if for all $q \in F$ the downset $\downarrow q$ is a chain (*i.e.*, a totally ordered set). A *leaf* is a maximal element of F . A *tree* is a forest with a bottom element, called the *root* of the tree. A *subforest* of a forest F is the downset of some $Q \subseteq F$. The *height* of a chain is the number of its elements. The *height* of a forest is the maximum height of any inclusion-maximal chain of the forest.

Let $\text{Sub}(F)$ denote the family of all subforests of a forest F . It so happens that $\text{Sub}(F)$ has a natural structure of Gödel algebra, where \wedge and \vee are given by union and intersection of subforests, and implication is defined, for

$F_1, F_2 \in \text{Sub}(F)$, as

$$F_1 \rightarrow F_2 = \{q \in F \mid \downarrow q \cap F_1 \subseteq \downarrow q \cap F_2\}.$$

The constants \perp, \top are the empty forest and F itself, respectively. Negation is defined by $\neg F_1 = F_1 \rightarrow \perp$. It turns out that any finite Gödel algebra is representable as $\text{Sub}(F)$, for some choice of F that is unique to within a poset isomorphism. See [9, §2] for a concise treatment and further references.

The forest \mathcal{F}_n such that $\mathcal{G}_n \cong \text{Sub}(\mathcal{F}_n)$ has special importance, as it is associated with the pure Gödel logic over the propositional variables X_1, \dots, X_n . We next show how to explicitly describe \mathcal{F}_n in the elementary language of $[0, 1]$ -valued assignments. This description plays a key role in what follows.

Definition 2.1 *We say that two assignments μ and ν are equivalent over the first n variables, or n -equivalent, written $\mu \equiv_n \nu$, if and only if there exists a permutation $\sigma : \underline{n} \rightarrow \underline{n}$ such that:*

$$0 \preceq_0 \mu(X_{\sigma(1)}) \preceq_1 \cdots \preceq_{n-1} \mu(X_{\sigma(n)}) \preceq_n 1, \quad (2.3)$$

$$0 \preceq_0 \nu(X_{\sigma(1)}) \preceq_1 \cdots \preceq_{n-1} \nu(X_{\sigma(n)}) \preceq_n 1,$$

where $\preceq_i \in \{<, =\}$, for $i = 0, \dots, n$.

Clearly, \equiv_n is an equivalence relation. Throughout, we write \mathcal{F}_n for the (finite) set of equivalence classes of \equiv_n . Here, we are abusing notation in that \mathcal{F}_n already denotes a forest such that $\mathcal{G}_n \cong \text{Sub}(\mathcal{F}_n)$. In fact, (i) in Proposition 2.4 below shows that our usage is harmless.

It is not difficult to show that if $\alpha(X_1, \dots, X_n)$ is a formula in Gödel logic, and μ, ν are two n -equivalent assignments, then

$$\mu(\alpha(X_1, \dots, X_n)) = 1 \text{ if and only if } \nu(\alpha(X_1, \dots, X_n)) = 1. \quad (2.4)$$

We can further endow \mathcal{F}_n with a partial order.

Definition 2.2 *Let $[\mu]_{\equiv_n}, [\nu]_{\equiv_n} \in \mathcal{F}_n$, and let $\sigma : \underline{n} \rightarrow \underline{n}$ be a permutation such that*

$$0 \preceq_0 \nu(X_{\sigma(1)}) \preceq_1 \cdots \preceq_{n-1} \nu(X_{\sigma(n)}) \preceq_n 1,$$

$$0 \tilde{\preceq}_0 \mu(X_{\sigma(1)}) \tilde{\preceq}_1 \cdots \tilde{\preceq}_{n-1} \mu(X_{\sigma(n)}) \tilde{\preceq}_n 1,$$

where $\preceq_i, \tilde{\preceq}_i \in \{<, =\}$, for $i = 0, \dots, n$. We define $[\mu]_{\equiv_n} \leq [\nu]_{\equiv_n}$ if and only if there exists an index $k \in \{0, \dots, n\}$ such that

i) $\tilde{\preceq}_i$ coincides with \preceq_i if $0 \leq i \leq k$,

ii) $\tilde{\preceq}_i$ coincides with $=$ if $k+1 \leq i \leq n$.

Example 2.3 *Let μ, ν, ξ be assignments such that*

- $\mu(X_1) = 1, \mu(X_2) = 1/3, \mu(X_3) = 0, \mu(X_4) = 1,$
- $\nu(X_1) = 1, \nu(X_2) = 1/4, \nu(X_3) = 0, \nu(X_4) = 1/2,$
- $\xi(X_1) = 1, \xi(X_2) = 1/2, \xi(X_3) = 0, \xi(X_4) = 1/2.$

For $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 4, \sigma(4) = 1,$ one has

- $0 = \mu(X_3) < \mu(X_2) < \mu(X_4) = \mu(X_1) = 1,$
- $0 = \nu(X_3) < \nu(X_2) < \nu(X_4) < \nu(X_1) = 1,$
- $0 = \xi(X_3) < \xi(X_2) = \xi(X_4) < \xi(X_1) = 1.$

Thus, according to Definition 2.2, $[\mu]_{\equiv_n} \leq [\nu]_{\equiv_n},$ and $[\xi]_{\equiv_n}$ is incomparable to both $[\mu]_{\equiv_n}$ and $[\nu]_{\equiv_n}.$

One checks that \leq in Definition 2.2 indeed is a partial order on $\mathcal{F}_n,$ and (\mathcal{F}_n, \leq) is in fact a forest [10, Lemma 3.3]. Direct inspection shows that

- a) the roots of the trees are the equivalence classes of Boolean assignments,
- b) the equivalence class $[\mu]_{\equiv_n}$ such that $\mu(X_1) = \dots = \mu(X_n) = 0$ is the only tree having height 1, and
- c) the leaves are those equivalence classes of assignments in which no variable is set to 1.

We can now sum up the relationships between finite forests and finite Gödel algebras, as follows.

For each $i = 1, \dots, n,$ let $\chi_i = \{[\mu]_{\equiv_n} \mid \mu(X_i) = 1\}$ be the i^{th} generating subforest of $\mathcal{F}_n.$ We recall that the prime (lattice) filters of a Gödel algebra G represent precisely those congruences Θ such that G/Θ is totally ordered.

Proposition 2.4 *Fix an integer $k \geq 0.$ (i) $\text{Sub}(\mathcal{F}_k)$ is (isomorphic to) the free Gödel algebra on k free generators. A free generating set is given by the collection of generating subforests. (ii) Up to isomorphism, the quotients of $\text{Sub}(\mathcal{F}_k)$ are precisely the algebras of the form $\text{Sub}(F),$ for $F \in \text{Sub}(\mathcal{F}_k).$ (iii) The set of prime filters ordered by reverse inclusion of $\text{Sub}(F)$ is order-isomorphic to F for every $F \in \text{Sub}(\mathcal{F}_k).$*

PROOF. The proof is a straightforward translation of [9, Remark 2 and Proposition 2.4] in the language of equivalence classes of assignments introduced above.

Figure 3 shows the forest $\mathcal{F}_2,$ whose nodes are labelled by the ordering of variables under a given assignment as in (2.3). However, for the sake of readability, here and in the following figure we write X_i instead of $\mu(X_i).$

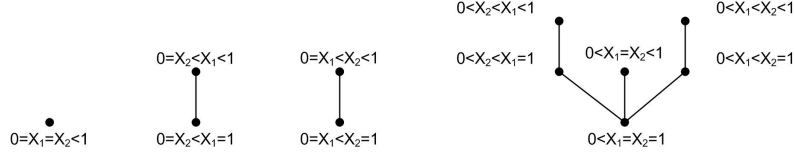


Fig. 3. The forest \mathcal{F}_2 .

2.3 The forest determined by P

We can now associate with P a uniquely determined forest. As an immediate consequence of Proposition 2.4, we can reformulate (2.1) as follows: P uniquely determines a congruence Θ' on $\text{Sub}(\mathcal{F}_n)$, and a subforest $F(P)$ of \mathcal{F}_n such that

$$\text{Sub}(\mathcal{F}_n)/\Theta' \cong \text{Sub}(F(P)) \cong \mathcal{G}(P).$$

To relate Θ' with the formula α_P in (2.2) or, equivalently, with $F(P)$, we shall give an explicit description of $F(P)$. To this end, it is convenient to introduce the following notion.

Definition 2.5 Let $[\mu]_{\equiv_n} \in \mathcal{F}_n$ and $x \in [0, 1]$. We say $[\mu]_{\equiv_n}$ is realized by P at x if there exists a permutation $\sigma : \underline{n} \rightarrow \underline{n}$ such that

$$0 \preceq_0 f_{\sigma(1)}(x) \preceq_1 \cdots \preceq_{n-1} f_{\sigma(n)}(x) \preceq_n 1,$$

$$0 \preceq_0 \mu(X_{\sigma(1)}) \preceq_1 \cdots \preceq_{n-1} \mu(X_{\sigma(n)}) \preceq_n 1,$$

where $\preceq_i \in \{<, =\}$, $i \in \{0, \dots, n\}$.

Proposition 2.6 We have

$$F(P) = \downarrow \{[\mu]_{\equiv_n} \in \mathcal{F}_n \mid [\mu]_{\equiv_n} \text{ is realized by } P \text{ at some } x \in [0, 1]\}. \quad (2.5)$$

PROOF. We first construct a subdirect representation of $\mathcal{G}(P)$. We shall then use Proposition 2.4 to identify $F(P)$ with the forest of prime filters of $\mathcal{G}(P)$. This will allow us to prove the desired equality (2.5).

To construct the subdirect representation, note that there exists a finite set $\{x_1, \dots, x_m\} \subseteq [0, 1]$ such that for each $y \in [0, 1]$, if $[\mu]_{\equiv_n} \in F(P)$ is realized by P at y , then it is also realized by P at x_i , for some $i \in \underline{m}$. Moreover, one checks that evaluating the elements of $\mathcal{G}(P)$ at x_i yields a totally ordered Gödel algebra C_{x_i} that is a homomorphic image of $\mathcal{G}(P)$ via the quotient map q_i given by restriction to x_i . The homomorphism

$$s : \mathcal{G}(P) \hookrightarrow \prod_{i=1}^m C_{x_i}$$

given by

$$g \in \mathcal{G}(P) \longmapsto (q_1(g), \dots, q_m(g))$$

is injective. Indeed, let $g \neq h \in \mathcal{G}(P)$, say $g(y) > h(y)$ for $y \in [0, 1]$. For the sake of brevity, we shall only deal with the case $1 > g(y) > h(y) > 0$. Then $g(y) = f_i(y)$ and $h(y) = f_j(y)$ for $i \neq j$. Let $[\mu]_{\equiv_n}$ be the assignment realized by P at y . There exists $u \in \underline{m}$ such that $[\mu]_{\equiv_n}$ is realized by P at x_u , and therefore $f_i(x_u) > f_j(x_u)$, which proves $s(g) \neq s(h)$.

It now follows that s is a subdirect representation of $\mathcal{G}(P)$. By Proposition 2.4(iii) we identify prime filters of $\mathcal{G}(P)$ with elements of $F(P) \subseteq \mathcal{F}_n$. The prime filters that are kernels of q_1, \dots, q_m must comprise all inclusion-minimal prime filters of $\mathcal{G}(P)$, *i.e.*, all leaves of $F(P)$, for otherwise s could not be a subdirect representation. Therefore, the classes $[\mu]_{\equiv_n}$ realized by P at some $x \in [0, 1]$ comprise all leaves of $F(P)$ (and possibly other elements). Since any forest is the downset of its leaves the proposition is proved.

In general, we associate with a formula $\alpha(X_1, \dots, X_n)$ the uniquely determined subforest of \mathcal{F}_n , denoted F_α , as follows:

$$F_\alpha = \{[\mu]_{\equiv_n} \in \mathcal{F}_n \mid \mu(\alpha) = 1\} . \quad (2.6)$$

By (2.4), F_α does not depend on the choice of μ . Clearly, F_α corresponds to the quotient algebra $\text{Sub}(\mathcal{F}_n)/\Theta'$, where Θ' is the congruence generated by $\alpha(X_1, \dots, X_n) = \top$. Finally, by the foregoing we have

$$F_{\alpha_P} = F(P) . \quad (2.7)$$

2.4 Finite-valued Gödel logics

In Section 4 we are going to deal with four-valued Gödel logic. Here we provide the needed background. Fix an integer $t \geq 2$, and consider the set of truth values $T_t = \{0 = \frac{0}{t-1}, \frac{1}{t-1}, \dots, \frac{t-2}{t-1}, \frac{t-1}{t-1} = 1\} \subseteq [0, 1]$. To define n -valued Gödel logic semantically, we consider the same set of well-formed formulas over X_1, X_2, \dots as for \mathbb{G}_∞ , but we restrict assignments to those taking values in T_t , that is, to *t-valued assignments*. A tautology of t -valued Gödel logic \mathbb{G}_t is defined as a formula that takes value 1 under any t -valued assignment. Syntactically, we need to add one axiom scheme to those of \mathbb{G}_∞ in order to obtain a completeness theorem for \mathbb{G}_t . Namely, consider the axiom

$$\alpha_1 \vee (\alpha_1 \rightarrow \alpha_2) \vee \dots \vee (\alpha_1 \wedge \dots \wedge \alpha_{t-1} \rightarrow \alpha_t) . \quad (\text{LIN}_t)$$

Using *modus ponens* as the only deduction rule, one proves that the axioms of \mathbb{G}_∞ together with (LIN_t) provide a complete axiomatization² of \mathbb{G}_t . We write

² Readers interested in proof-theoretic aspects of Gödel logics are referred to [11] for an extensive discussion with further references.

$\vdash_{\mathbb{G}_t} \alpha$ to mean that the formula α is provable in \mathbb{G}_t .

It is straightforward to extend to \mathbb{G}_t the combinatorial representation theory of Subsection 2.2. For this, we use partially ordered equivalence classes of assignments as in Definitions 2.1 and 2.2, except that we only consider t -valued assignments. Contemplation of the meaning of (LIN_t) shows that the forest \mathcal{F}_n^t associated with the pure t -valued Gödel logic \mathbb{G}_t is order-isomorphic to the subforest of \mathcal{F}_n consisting of all elements having height at most $t-1$. In other words, truncating \mathcal{F}_n to height $t-1$ yields \mathcal{F}_n^t . The correspondence for \mathbb{G}_∞ between subforests, formulas, and quotient algebras given by the foregoing now extends to \mathbb{G}_t in the obvious manner.

3 Gödel Approximation of Ruspini Partitions

Let P be a Ruspini partition. It is clear that those assignments μ to X_1, \dots, X_n such that either $\mu(X_i) = 0$, for all $i \in \underline{n}$, or $\mu(X_i) < 1$ for exactly one index i , and $\mu(X_j) = 0$, for all $j \neq i$, cannot evaluate α_P to 1. Equivalently, these assignments cannot be realized by P at any $x \in [0, 1]$. The following definition isolates a class of subforests $\mathcal{R}_n \subseteq \mathcal{F}_n$ that omits from \mathcal{F}_n precisely those points corresponding to such assignments.

Definition 3.1 *We denote by \mathcal{R}_n the subforest of \mathcal{F}_n obtained by removing from \mathcal{F}_n the single tree having height 1, and the leaves of all the trees having height 2. We call \mathcal{R}_n the Ruspini forest.*

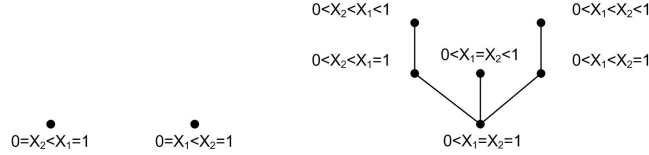


Fig. 4. The Ruspini forest \mathcal{R}_2 .

We now show how to explicitly axiomatize \mathcal{R}_n .

Definition 3.2 *We define the Ruspini axiom $\rho_n = \alpha \vee \beta$, where*

$$\alpha = \bigvee_{1 \leq i < j \leq n} (\neg X_i \wedge \neg X_j), \quad \text{and} \quad \beta = \bigvee_{1 \leq i \leq n} (X_i \wedge \bigwedge_{1 \leq j \neq i \leq n} \neg X_j).$$

Recall that the formula ρ_n uniquely determines a subforest $F_{\rho_n} \subseteq \mathcal{F}_n$ as in (2.6). In fact:

Proposition 3.3 $F_{\rho_n} = \mathcal{R}_n$.

PROOF. Fix an assignment μ . Since

$$\mu(\neg\neg X) = \begin{cases} 0 & \text{if } \mu(X) = 0 \\ 1 & \text{otherwise,} \end{cases}$$

$\mu(\alpha) \neq 1$ if and only if at most one variable X_{i_0} satisfies $\mu(X_{i_0}) \neq 0$.

Observe now that $\mu(\beta) = 1$ if and only if there exists $i \in \underline{n}$ such that, for $j \neq i$, $\mu(X_i) = 1$ and $\mu(X_j) = 0$.

Therefore, $\mu(\rho_n) = \mu(\alpha \vee \beta) \neq 1$ if and only if there exists $i_0 \in \underline{n}$ such that, for $j \neq i_0$, $\mu(X_{i_0}) < 1$ and $\mu(X_j) = 0$. It is now straightforward to verify that the latter condition holds if and only if $[\mu]_{\equiv_n} \notin \mathcal{R}_n$.

Let us introduce a property of P that we shall use in our main result. Let $\lambda : [0, 1] \rightarrow [0, 1]$ be an order preserving map such that $\lambda(0) = 0$ and $\lambda(1) = 1$, and let $t = \inf \lambda^{-1}(1)$. If the restriction of λ to $[0, t]$ is an order isomorphism between $[0, t]$ and $[0, 1]$, we say λ is a *comparison map*.

Definition 3.4 We say P is a weak Ruspini partition if for all $x \in [0, 1]$, there exist $y \in [0, 1]$, a comparison map λ , and an order isomorphism γ from $[0, 1]$ to itself, such that

- (i) $\lambda(f_i(y)) = f_i(x)$, for all $i \in \underline{n}$.
- (ii) $\sum_{i=1}^n \gamma(f_i(y)) = 1$.

Example 3.5 The set of functions $P = \{f_1, f_2\}$ shown in Figure 5 is a weak Ruspini partition. Indeed, for $x = 0$ or $x = 1$, conditions (i) and (ii) in

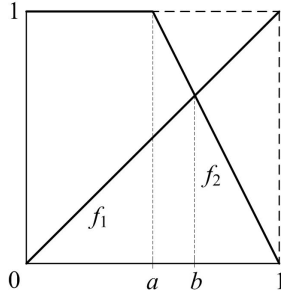


Fig. 5. A weak Ruspini partition $\{f_1, f_2\}$.

Definition 3.4 are trivially satisfied with $y = x$, and λ and γ the identity functions. (More generally, for all $x \in [0, 1]$ where the Ruspini condition locally holds as $\sum_{i=1}^n f_i(x) = 1$, (i) and (ii) are satisfied in this manner.) If $x \in (a, 1)$ we can still choose $y = x$ and λ the identity function. Then, since the values of f_1, f_2 at x satisfy $0 < f_1(x), f_2(x) < 1$, it is clear that there is an order isomorphism γ that shifts these values to $0 < \gamma(f_1(x)), \gamma(f_2(x)) < 1$ so that

$\gamma(f_1(x)) + \gamma(f_2(x)) = 1$. Finally, consider $x \in (0, a]$. Here, $x = y$ does not work, because then $0 < f_1(x) < f_2(x) = 1$, and regardless of our choice of γ we have $0 < \gamma(f_1(x)) < \gamma(f_2(x)) = 1$, whence $\gamma(f_1(x)) + \gamma(f_2(x)) > 1$. However, let $y \in (a, b)$. Then, we can choose λ such that $\lambda(f_1(y)) = f_1(x)$, and the restriction of λ to $[0, f_2(y)]$ is an order isomorphism onto $[0, 1]$ —hence $\lambda(f_2(y)) = f_2(x)$, too. In particular λ carries $[f_2(y), 1]$ to 1. For this choice of λ , (i) is satisfied. As before, it is easy to construct an order isomorphism γ satisfying (ii) with respect to our chosen y . (Thus, λ preserves the relative order of the values of f_i , except that it can collapse the values above a fixed $t \in (0, 1]$ to 1. Then the Ruspini condition is to be satisfied by the values at y .)

The importance of comparison maps to our purposes is brought out by our next result. The following lemma relates the order between points of \mathcal{F}_n realized by any P with the existence of an appropriate comparison map. Further, it relates the existence of leaves of \mathcal{R}_n realized by P with the existence of an appropriate order isomorphism of the real unit interval.

Lemma 3.6 *Let $[\mu]_{\equiv_n}, [\nu]_{\equiv_n} \in \mathcal{F}_n$ and $x, y \in [0, 1]$ such that $[\mu]_{\equiv_n}$ and $[\nu]_{\equiv_n}$ are realized by P at x and y , respectively. Then the following are equivalent.*

- (i) $[\mu]_{\equiv_n} \leq [\nu]_{\equiv_n}$.
- (ii) *There exists a comparison map $\lambda : [0, 1] \rightarrow [0, 1]$ with $\lambda(f_i(y)) = f_i(x)$, for all $i \in \underline{n}$.*

Moreover, the following are equivalent.

- (iii) $[\mu]_{\equiv_n}$ is a leaf of \mathcal{R}_n .
- (iv) *There exists an order isomorphism $\gamma : [0, 1] \rightarrow [0, 1]$ with $\sum_{i=1}^n \gamma(f_i(x)) = 1$.*

PROOF. (i) \Rightarrow (ii). By Definitions 2.2 and 2.5, there exists a permutation $\sigma : \underline{n} \rightarrow \underline{n}$ such that

$$\begin{aligned} 0 &\preceq_0 f_{\sigma(1)}(y) \preceq_1 \cdots \preceq_{n-1} f_{\sigma(n)}(y) \preceq_n 1, \\ 0 &\tilde{\preceq}_0 f_{\sigma(1)}(x) \tilde{\preceq}_1 \cdots \tilde{\preceq}_{n-1} f_{\sigma(n)}(x) \tilde{\preceq}_n 1, \end{aligned}$$

where $\preceq_i, \tilde{\preceq}_i \in \{<, =\}$, and there is $k \in \{0, \dots, n\}$ satisfying (i) and (ii) in Definition 2.2. We deal with the case $k < n$ only; the case $k = n$ is a trivial variation thereof. We define Λ by $\Lambda(f_{\sigma(i)}(y)) = f_{\sigma(i)}(x)$, for $1 \leq i \leq k$, and $\Lambda(f_{\sigma(i)}(y)) = 1$ if $k+1 \leq i \leq n$. We extend Λ to a comparison map as follows. Consider the closed intervals $I_0 = [0, f_{\sigma(1)}(y)]$, $J_0 = [0, f_{\sigma(1)}(x)]$, $I_i = [f_{\sigma(i)}(y), f_{\sigma(i+1)}(y)]$ and $J_i = [f_{\sigma(i)}(x), f_{\sigma(i+1)}(x)]$, for $1 \leq i \leq k$. Now let us fix $0 \leq h \leq k$. Note that if I_h collapses to a point, then J_h also collapses to a point. Therefore, in all cases we can choose order isomorphisms $\lambda_h : I_h \rightarrow J_h$. Moreover, set $I_{k+1} = [f_{\sigma(k+1)}(y), 1]$ and $\lambda_{k+1} : I_{k+1} \rightarrow \{1\}$. Since λ_h and λ_{h+1} agree at $I_h \cap I_{h+1}$ by construction, the function $\lambda : [0, 1] \rightarrow [0, 1]$ defined by

$\lambda(r) = \lambda_j(r)$ if $r \in I_j$, for $0 \leq j \leq k+1$, is a comparison map satisfying (ii).

(ii) \Rightarrow (i). Immediate from Definitions 2.2 and 2.5.

(iii) \Rightarrow (iv). It is an exercise to check that $[\mu]_{\equiv_n}$ is a leaf of \mathcal{R}_n if and only if exactly one of the following two cases hold.

Case 1. There exists i_0 such that $\mu(X_{i_0}) = 1$ and $\mu(X_i) = 0$ for $i \neq i_0$.
Let γ be the identity map. By Definition 2.5, we have $\sum_{i=1}^n \gamma(f_i(x)) = 1$.

Case 2. For all i , $\mu(X_i) < 1$, and there exist i_0, i_1 such that $0 < \mu(X_{i_0}) \leq \mu(X_{i_1})$.

Let us write

$$0 \preceq_0 f_{\sigma(1)}(x) \preceq_1 \cdots \preceq_{n-1} f_{\sigma(n)}(x) \preceq_n 1,$$

for some permutation σ and $\preceq_i \in \{<, =\}$. We shall assume \preceq_0 is $<$. The case where some f_i takes value zero at x is entirely similar.

Now consider the $(n-1)$ -dimensional simplex³ S_n , given by the convex hull of the standard basis of \mathbb{R}^n . Let $S_n^{(1)}$ be the simplicial complex given by the first barycentric subdivision of S_n . The $(n-1)$ -dimensional simplices of $S_n^{(1)}$ are in bijection with the permutations of \underline{n} , and the solution set of the inequalities

$$0 \leq r_1 \leq \cdots \leq r_n \leq 1 \tag{3.1}$$

in S^n is an $(n-1)$ -dimensional simplex $S \in S_n^{(1)}$. Consider the equalities

$$r_i = r_{i+1} \tag{3.2}$$

for each $i = 1, \dots, n-1$ such that \preceq_i is $=$. Then the solution set of (3.1) and (3.2) is a nonempty face T of S . Consider next the strict inequalities

$$\begin{cases} r_i < r_{i+1} \\ 0 < r_1 \\ r_n < 1 \end{cases} \tag{3.3}$$

for all $i = 1, \dots, n-1$ such that \preceq_i is $<$. Then the solution set of (3.1), (3.2), and (3.3) is the relative interior T° of T . Since T is nonempty, T° is nonempty. The barycenter $b = (b_1, \dots, b_n)$ of T lies in T° . Since $b \in S_n$, we have $\sum_{k=1}^n b_k = 1$. Moreover, by construction,

$$0 \preceq_0 b_1 \preceq_1 \cdots \preceq_{n-1} b_n \preceq_n 1.$$

We define Γ by $\Gamma(f_{\sigma(i)}) = b_i$. Arguing as in the proof of (i) \Rightarrow (ii), we conclude that there is an extension of Γ to an order isomorphism $\gamma : [0, 1] \rightarrow [0, 1]$ satisfying (iv).

³ For all unexplained notions in combinatorial topology, please see [12].

(iv) \Rightarrow (iii). Suppose $[\mu]_{\equiv_n}$ is not a leaf of \mathcal{R}_n . Thus, exactly one of the following two cases holds.

Case 1. $[\mu]_{\equiv_n} \in \mathcal{F}_n \setminus \mathcal{R}_n$.

In this case there exists i_0 such that $\mu(X_{i_0}) < 1$ and $\mu(X_i) = 0$ for $i \neq i_0$. Using Definition 2.5, we have $\sum_{i=1}^n \gamma(f_i(x)) < 1$, for each order isomorphism γ .

Case 2. $[\mu]_{\equiv_n} \in \mathcal{R}_n$, but $[\mu]_{\equiv_n} \in \mathcal{R}_n$ is not a leaf of \mathcal{R}_n .

It is easy to check that there exist i_0, i_1 such that $0 < \mu(X_{i_0}) \leq \mu(X_{i_1}) = 1$. Using Definition 2.5, we have $f_{i_1}(x) = 1$ and $f_{i_0}(x) > 0$, and thus $\sum_{i=1}^n \gamma(f_i(x)) > 1$, for each order isomorphism γ .

To state our main result we still need to show how to obtain a formula $\psi_{[\mu]_{\equiv_n}}$ associated with a given element $[\mu]_{\equiv_n} \in \mathcal{F}_n$ such that $\psi_{[\mu]_{\equiv_n}}$ evaluates to 1 exactly on $\downarrow [\mu]_{\equiv_n}$. To this end, we define the derived connective $\alpha \triangleleft \beta = ((\beta \rightarrow \alpha) \rightarrow \beta)$. Given an assignment μ we have that

$$\mu(\alpha \triangleleft \beta) = \begin{cases} 1 & \text{if } \mu(\alpha) < \mu(\beta) \text{ or } \mu(\alpha) = \mu(\beta) = 1 \\ \mu(\beta) & \text{otherwise.} \end{cases}$$

Suppose now that, for a given permutation $\sigma : \underline{n} \rightarrow \underline{n}$,

$$0 \preceq_0 \mu(X_{\sigma(1)}) \preceq_1 \cdots \preceq_{n-1} \mu(X_{\sigma(n)}) \preceq_n 1,$$

where $\preceq_i \in \{<, =\}$, $i = 0, \dots, n$. We associate to $[\mu]_{\equiv_n}$ the formula

$$\psi_{[\mu]_{\equiv_n}} = (\perp \bowtie_0 X_{\sigma(1)}) \wedge (X_{\sigma(1)} \bowtie_1 X_{\sigma(2)}) \wedge \cdots \wedge (X_{\sigma(n)} \bowtie_n \top), \quad (3.4)$$

where $\bowtie_i = \triangleleft$ if \preceq_i is $<$, and $\bowtie_i = \leftrightarrow$ otherwise.

Lemma 3.7 $F_{\psi_{[\mu]_{\equiv_n}}} = \downarrow [\mu]_{\equiv_n}$.

PROOF. We omit the straightforward verification. Compare [13], where a theory of *chain normal forms* for Gödel logic is introduced using similar tools.

Given a forest $F \subseteq \mathcal{F}_n$, let us indicate with $\text{Leaf}(F)$ the set of leaves of F .

Lemma 3.8 Fix a forest $F \subseteq \mathcal{F}_n$, and let $\alpha(X_1, \dots, X_n)$ be a formula as in (2.6) such that $F_\alpha = F$. Then

$$\vdash \alpha \leftrightarrow \bigvee_{l \in \text{Leaf}(F_\alpha)} \psi_l. \quad (3.5)$$

PROOF. Set $\beta = \bigvee_{l \in \text{Leaf}(F_\alpha)} \psi_l$. Then, by the definition of \leftrightarrow , (3.5) holds if and only if $F_\alpha = F_\beta$. But, by Lemma 3.7 along with the definition of \vee , F_β is the downset of the leaves of F_α , whence it coincides with F_α .

Note that, in particular, Lemma 3.8 yields the promised explicit construction of α_P , for any family of fuzzy sets P . Indeed, using (2.7),

$$\vdash \alpha_P \leftrightarrow \bigvee_{l \in \text{Leaf}(F(P))} \psi_l. \quad (3.6)$$

Definition 3.9 *We say that a forest F is a Ruspini subforest if $F \subseteq \mathcal{R}_n$ and each leaf of F is a leaf of \mathcal{R}_n .*

We can finally prove our first main result.⁴

Theorem 3.10 *For any choice of P the following are equivalent.*

- (i) P is a weak Ruspini partition.
- (ii) $F(P)$ is a Ruspini subforest.
- (iii) \mathbb{G}_∞ proves

$$\alpha_P \leftrightarrow \bigvee_{l \in \text{Leaf}(F_{\alpha_P}) \cap \text{Leaf}(\mathcal{R}_n)} \psi_l. \quad (3.7)$$

Moreover, for any Ruspini subforest F there exists a Ruspini partition $P' = \{f'_1, \dots, f'_n\}$, with $f'_i : [0, 1] \rightarrow [0, 1]$, such that $F(P') = F$.

PROOF. Recall from (2.7) that $F_{\alpha_P} = F(P)$. We tacitly use the latter identification in the proof below.

(i) \Rightarrow (ii). By Lemma 3.6, we can reformulate Definition 3.4 in terms of assignments as follows. For all $[\mu]_{\equiv_n} \in \mathcal{F}_n$ realized by P at some $x \in [0, 1]$, there exists $[\nu]_{\equiv_n} \geq [\mu]_{\equiv_n}$ realized by P at some $y \in [0, 1]$ such that $[\nu]_{\equiv_n}$ is a leaf of \mathcal{R}_n . Thus, by Proposition 2.6, $F(P)$ is exactly the downset of those leaves of \mathcal{R}_n realized by P at some $x \in [0, 1]$.

(ii) \Rightarrow (iii). By Definition 3.9, each leaf $k \in \text{Leaf}(F_{\alpha_P})$ is a leaf of \mathcal{R}_n . Hence, $\text{Leaf}(F_{\alpha_P}) \cap \text{Leaf}(\mathcal{R}_n) = \text{Leaf}(F_{\alpha_P})$, and the result follows from (3.6).

(iii) \Rightarrow (i). Suppose P is not a weak Ruspini partition. By Definition 3.4, using Lemma 3.6, there exists $k \in F_{\alpha_P}$ such that one of the following two conditions holds.

- (a) $k \in \mathcal{F}_n \setminus \mathcal{R}_n$.

⁴ In [6, p. 170] a different version of this theorem appears, where the formula in (iii) regrettably contains a mistake.

(b) $k \in \mathcal{R}_n$ is a maximal element of F_{α_P} , but it is not a leaf of \mathcal{R}_n .

We will show that both (a) and (b) lead to a contradiction. To this purpose, set $\beta = \bigvee_{l \in \text{Leaf}(F_{\alpha_P}) \cap \text{Leaf}(\mathcal{R}_n)} \psi_l$. Then F_β , the forest associated with β via (2.6), is a subforest of \mathcal{R}_n . Indeed, by Lemma 3.7, F_β is the downset of those $l \in \mathcal{F}_n$ satisfying $l \in \text{Leaf}(F_{\alpha_P}) \cap \text{Leaf}(\mathcal{R}_n)$. By the definition of \leftrightarrow , (3.7) holds if and only if $F_{\alpha_P} = F_\beta$. Suppose (a) holds. Then k is an element of F_{α_P} lying strictly above a leaf of \mathcal{R}_n . Since, as just shown, $F_\beta \subseteq \mathcal{R}_n$, we infer $F_{\alpha_P} \neq F_\beta$, a contradiction. Next suppose (b) holds. We claim $k \notin F_\beta$. Indeed, since F_β is the downset of those $l \in \mathcal{F}_n$ satisfying $l \in \text{Leaf}(F_{\alpha_P}) \cap \text{Leaf}(\mathcal{R}_n)$, $k \in F_\beta$ if and only if there exists such an l satisfying $l \geq k$. Since $k \notin \text{Leaf}(\mathcal{R}_n)$ by (b), we have $l > k$. Since $l \in \text{Leaf}(F_{\alpha_P})$, the latter inequality means that k is not a leaf of F_{α_P} , a contradiction. We conclude $k \notin F_\beta$, whence $F_{\alpha_P} \neq F_\beta$, as was to be shown.

Finally, we prove the last statement of the theorem. Let $[\mu_1]_{\equiv n}, \dots, [\mu_m]_{\equiv n}$ be the leaves of F . Partition the interval $[0, 1]$ into m intervals $I_1 = [0, x_1]$, $I_2 = (x_1, x_2], \dots, I_m = (x_{m-1}, 1 = x_m]$. We construct the functions f'_i as follows. For $i \in \underline{n}$, $j \in \underline{m}$, we set $f'_i(x) = C_{ij} \in \mathbb{R}$ if $x \in I_j$. The constants C_{ij} are chosen so that

- (c) $[\mu_j]_{\equiv n}$ is realized by P' at x_j ,
- (d) $\sum_{i=1}^n C_{ij} = 1$.

Obviously, it is always possible to choose C_{ij} so that (c) holds. The proof of (iii) \Rightarrow (iv) in Lemma 3.6 shows that, in fact, it is always possible to choose C_{ij} so that both (c) and (d) hold.

As a first corollary, we can count the number of Ruspini partitions with n fuzzy sets that can be told apart by Gödel logic. In [14, Theorem 3] it is shown that the number of leaves of \mathcal{F}_n is

$$L_n = 2 \sum_{k=1}^n k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}, \quad (3.8)$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the number of partitions of an n -element set into k classes, *i.e.*, the *Stirling number of the second kind*. The number $\sum_{k=1}^n k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the n^{th} *ordered Bell number*, *i.e.*, the number of all ordered partitions of \underline{n} . Compare sequence A000670 in [15].

Consider $P' = \{f'_1, \dots, f'_n\}$, where $f'_i : [0, 1] \rightarrow [0, 1]$. In the light of Section 2, let us say that P' is *Gödel-equivalent* to P if $F(P) = F(P')$, or, equivalently, $\vdash \alpha_P \leftrightarrow \alpha_{P'}$. Then:

Corollary 3.11 *The number of equivalence classes of Gödel-equivalent weak Ruspini partitions of n elements is $2^{L_n-1} - 1$, where L_n is given by (3.8).*

PROOF. A weak Ruspini partition P is characterized, up to Gödel-equivalence, by the forest $F(P)$, and therefore by a subset of leaves of \mathcal{R}_n . Noting that the number of leaves of \mathcal{R}_n is $L_n - 1$, and that for every weak Ruspini partition P , $F(P) \neq \emptyset$, the corollary follows.

Our second corollary deals with continuity. Since implication in Gödel logic has a discontinuous semantics, it is impossible to force continuity of all functions of a Ruspini partition (up to Gödel-equivalence). However, it is always possible to bound the number of discontinuities:

Corollary 3.12 *(i) There is a Ruspini subforest F such that whenever $F(P) = F$ then each $f_i \in P$ has a point of discontinuity. (ii) For all Ruspini subforests F with L leaves there is a choice of a Ruspini partition $P' = \{f'_1, \dots, f'_n\}$, with $F(P') = F$ such that each $f'_i : [0, 1] \rightarrow [0, 1]$ has at most $L - 1$ points of discontinuity.*

PROOF. (i) It suffices to choose $F \subseteq \mathcal{R}_n$ as the forest of all Boolean assignments which are leaves of \mathcal{R}_n . (ii) The construction used in the proof of the last statement of Theorem 3.10 yields the desired P' .

4 Four-valued Gödel Logic, and 2-overlapping Ruspini Partitions

Following the same outline of the previous section, we now investigate how Gödel logic expresses the 2-overlapping property of the family P of fuzzy sets.

Definition 4.1 *We denote by \mathcal{T}_n the subforest of \mathcal{F}_n obtained by removing from \mathcal{F}_n all the trees of height > 3 .*

Remark 4.2 *\mathcal{T}_n is the subforest of all equivalence classes of assignments $[\mu]_{\equiv_n} \in \mathcal{F}_n$ such that for all $i \neq j \neq k \in \underline{n}$, at least one of $\mu(X_i) = 0$, $\mu(X_j) = 0$, $\mu(X_k) = 0$, holds.*

We can immediately show how to axiomatize \mathcal{T}_n .

Definition 4.3 *We define the 2-overlapping axiom τ_n by*

$$\tau_n = \bigwedge_{1 \leq i < j < k \leq n} \neg(X_i \wedge X_j \wedge X_k).$$

Lemma 4.4 $F_{\tau_n} = \mathcal{T}_n$.

PROOF. Fix an assignment μ . Note that $\mu(\tau_n) \neq 1$ if and only if there exist $i \neq j \neq k \in \underline{n}$ such that $\mu(X_i) > 0$, $\mu(X_j) > 0$, and $\mu(X_k) > 0$. It is now straightforward to verify that the latter condition holds if and only if $[\mu]_{\equiv_n} \notin \mathcal{T}_n$.

Lemma 4.5 For any choice of P ,

$$\vdash_{\mathbb{G}_4} \alpha_P \rightarrow \tau_n \quad \text{if and only if} \quad \vdash \alpha_P \rightarrow \tau_n.$$

PROOF. (\Leftarrow) Trivial.

(\Rightarrow) The formula $\alpha_P \rightarrow \tau_n$ is a tautology of \mathbb{G}_4 if and only if

$$F_{\alpha_P} \cap \mathcal{F}_n^4 \subseteq F_{\tau_n} \cap \mathcal{F}_n^4. \quad (4.1)$$

Since, by Lemma 4.4, $F_{\tau_n} = \mathcal{T}_n$, and since $\mathcal{T}_n \subseteq \mathcal{F}_n^4$ by direct inspection, Condition (4.1) is equivalent to

$$F_{\alpha_P} \cap \mathcal{F}_n^4 \subseteq \mathcal{T}_n. \quad (4.2)$$

We show $F_{\alpha_P} \subseteq \mathcal{T}_n$. Suppose there exists $[\mu]_{\equiv_n} \in F_{\alpha_P}$ such that $[\mu]_{\equiv_n} \notin \mathcal{T}_n$ (*absurdum hypothesis*). By (4.2) we have $[\mu]_{\equiv_n} \in F_{\alpha_P} \setminus \mathcal{F}_n^4$. Therefore, the class $[\mu]_{\equiv_n}$ must belong to a tree of \mathcal{F}_n^4 of height > 3 . If $[\nu]_{\equiv_n}$ is the root of such tree, we have $[\nu]_{\equiv_n} \in F_{\alpha_P} \cap \mathcal{F}_n^4$ but, by Definition 4.1, $[\nu]_{\equiv_n} \notin \mathcal{T}_n$. This contradicts (4.2), and our claim is settled. Thus, $F_{\alpha_P} \subseteq \mathcal{T}_n$ and the formula $\alpha_P \rightarrow \tau_n$ is a tautology of \mathbb{G}_∞ .

Using the preceding lemma, we can now prove:

Theorem 4.6 For any choice of P , the following are equivalent.

- (i) P is a 2-overlapping family.
- (ii) $F(P)$ is a subforest of \mathcal{T}_n .
- (iii) $\vdash_{\mathbb{G}_4} \alpha_P \rightarrow \tau_n$.

PROOF. (i) \Rightarrow (ii). All $[\mu]_{\equiv_n} \in \mathcal{F}_n^4$ realized by P at some $x \in [0, 1]$ are such that for all $i \neq j \neq k \in \underline{n}$, at least one of $\mu(X_i) = 0$, $\mu(X_j) = 0$, $\mu(X_k) = 0$, holds. Thus, $F(P)$ is subforest of \mathcal{T}_n .

(ii) \Rightarrow (iii). Since $F(P) \subseteq \mathcal{T}_n$, we have $\vdash \alpha_P \rightarrow \tau_n$. But then $\vdash_{\mathbb{G}_4} \alpha_P \rightarrow \tau_n$.

(iii) \Rightarrow (i). Suppose P is not a 2-overlapping family (*absurdum hypothesis*). In other words, suppose that there exist $i \neq j \neq k \in \underline{n}$, and $x \in [0, 1]$, such that $f_i(x) > 0$, $f_j(x) > 0$, and $f_k(x) > 0$. Thus, there exists $[\mu]_{\equiv_n} \in \mathcal{F}_n$ realized by P at x , such that $\mu(X_i) > 0$, $\mu(X_j) > 0$, and $\mu(X_k) > 0$. Using (2.7), $[\mu]_{\equiv_n} \in F_{\alpha_P}$. Clearly, $[\mu]_{\equiv_n} \notin F_{\tau_n}$. Therefore, $\alpha_P \rightarrow \tau_n$ is not a tautology of \mathbb{G}_∞ . By Lemma 4.5, $\alpha_P \rightarrow \tau_n$ is not a tautology of \mathbb{G}_4 . This contradicts (iii) and completes the proof.

Our final aim is to combine Theorems 3.10 and 4.6, that is, to axiomatize 2-overlapping weak Ruspini partitions in four-valued Gödel logic.

Theorem 4.7 *For any choice of P the following are equivalent.*

- (i) P is a 2-overlapping weak Ruspini partition.
- (ii) $F(P)$ is a Ruspini subforest contained in \mathcal{T}_n .
- (iii) $\vdash_{\mathbb{G}_4} \alpha \wedge \beta$, where

$$\alpha = \alpha_P \leftrightarrow \bigvee_{l \in \text{Leaf}(F_{\alpha_P}) \cap \text{Leaf}(\mathcal{R}_n)} \psi_l, \quad (4.3)$$

$$\beta = (\alpha_P \rightarrow \tau_n).$$

Moreover, for any Ruspini subforest F contained in \mathcal{T}_n there exists a 2-overlapping Ruspini partition $P' = \{f'_1, \dots, f'_n\}$, with $f'_i : [0, 1] \rightarrow [0, 1]$, such that $F(P') = F$.

PROOF. (i) \Rightarrow (ii). By Theorem 3.10, $F(P)$ is a Ruspini subforest. By Theorem 4.6, $F(P) \subseteq \mathcal{T}_n$.

(ii) \Rightarrow (iii). By Theorem 3.10, the formula α is a tautology of \mathbb{G}_∞ , and thus a tautology of \mathbb{G}_4 . By Theorem 4.6, the formula β is a tautology of \mathbb{G}_4 . Therefore, the formula $\alpha \wedge \beta$ is a tautology of \mathbb{G}_4 .

(iii) \Rightarrow (i). Since $\alpha \wedge \beta$ is a tautology of \mathbb{G}_4 , we have $\vdash_{\mathbb{G}_4} \beta$. By Theorem 4.6, P is 2-overlapping. Moreover, by Lemma 4.5, $\vdash_{\mathbb{G}_4} \beta$ implies $\vdash \beta$, and therefore

$$F_{\alpha_P} \subseteq \mathcal{T}_n. \quad (4.4)$$

It remains to show that P is a weak Ruspini partition. The argument is analogous to that in (iii) \Rightarrow (i) of Theorem 3.10. Indeed, we note that \mathbb{G}_4 proves the formula $\alpha \leftrightarrow \beta$ if and only if $F_\alpha \cap \mathcal{F}_n^4 = F_\beta \cap \mathcal{F}_n^4$. Moreover, by (4.4), the element k appearing in conditions (a) and (b) in the proof of Theorem 3.10 necessarily belongs to τ_n , and thus to \mathcal{F}_n^4 .

The last statement of the theorem is an immediate consequence of Theorems 3.10 and 4.6.

The analogue of Corollary 3.11 for 2-overlapping weak Ruspini partition is as follows.

Corollary 4.8 *The number of classes of Gödel-equivalent 2-overlapping weak Ruspini partitions of n elements is $2^{\frac{3n^2-n}{2}} - 1$.*

PROOF. A 2-overlapping weak Ruspini partition P is characterized, up to Gödel equivalence, by the forest $F(P)$, and therefore by a subset of leaves of $\mathcal{R}_n \cap \mathcal{T}_n$. We observe that, by Definitions 3.1 and 4.1, $\mathcal{R}_n \cap \mathcal{T}_n$ is the forest obtained by removing from \mathcal{T}_n the single tree having height 1, and the leaves of all the trees having height 2. Thus, $\mathcal{R}_n \cap \mathcal{T}_n$ contains exactly $\binom{n}{1}$ forests of height 1, and $\binom{n}{2}$ forests of height 3. Since the trees of height 3 have precisely 3 leaves, the total number of leaves of $\mathcal{R}_n \cap \mathcal{T}_n$ is

$$\binom{n}{1} + 3\binom{n}{2} = \frac{3n^2 - n}{2}.$$

Noting that, for every 2-overlapping weak Ruspini partition P , $F(P) \neq \emptyset$, the corollary follows.

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