A Characterisation of Bases of Triangular Fuzzy Sets

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***Abstract*— Fuzzy sets featuring in applications to fuzzy control systems are often required to satisfy specific conditions such as, e.g., convexity or normality. In the same connection, a widespread choice is to work with fuzzy sets whose graphs have triangular shape. The purpose of this paper is to show that the former conditions may be regarded as attempts at approximating the latter choice. Specifically, as our main result we prove that a reasonable set of such conditions suffices to characterise families of triangular fuzzy sets. A second result provides an additional characterisation of such families in terms of properties of the curve that they parametrise.**

I. Introduction, and discussion of results

Let $[0,1]$ be the real unit interval. In this paper, by a *fuzzy set* we shall mean a *continuous* function $f\colon [0,1] \to [0,1]$. Throughout, we fix a positive integer n, and a non-empty family

$$P = \{f_1, \dots, f_n\}$$

of fuzzy sets. We write $\underline{n}$ for the set $\{1, \dots, n\}$.

Often, fuzzy sets are required to satisfy additional conditions that are deemed useful for the specific application under consideration. Here is a popular one that is usually traced back to [1, p. 28]. We say P is a *Ruspini partition* if for all $x \in [0,1]$

$$\sum_{i=1}^{n} f_i(x) = 1\,. \tag{1}$$

Figure 1 shows an example of a Ruspini partition. To

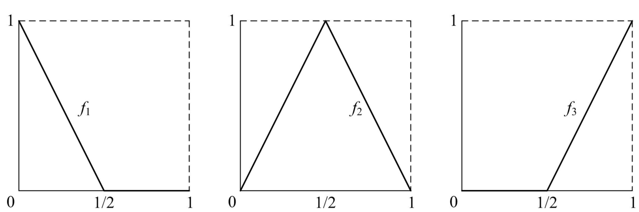

Fig. 1. A 2-overlapping Ruspini partition $\{f_1, f_2, f_3\}$.

indicate how this assumption is useful, suppose the values of the abscissa in Figure 1 represent normalised readings of an observable property of some physical system — say, temperature. Fuzzy sets, then, provide a means to attribute degrees of truth to propositions about the system temperature (with 0 denoting absolute truth and 1 denoting absolute falsity). For instance, the three sentences "The temperature is low", "The temperature is medium", and "The temperature is high" might be assigned the truth-value $f_1(t)$, $f_2(t)$, and $f_3(t)$, respectively, whenever t is the normalised temperature of the system. Insofar as addition of truth-degrees can be interpreted as a (non-idempotent) generalisation of logical disjunction, then, one sees that the Ruspini condition may be regarded as an exhaustiveness requirement: for any given temperature t, it is always true that the temperature is either low, or medium, or high.[1]

Beside (1), there are several other obvious properties satisfied by the fuzzy sets in Figure 1. For instance, fuzzy system designers often prefer fuzzy sets whose graphs overlap with at most one neighbouring graph to the left, and one to the right, as in Figure 1. In our example, this restriction amounts to saying that it is always false that the temperature is at the same time low, medium, and high — no matter what the observed value t is. By contrast, one may wish to allow configurations such as the one in Figure 2, where the same temperature may at the same time be considered low, medium, and high (to possibly different degrees). While this is sometimes called for by specific situations, it turns out that membership functions are most often chosen so as to keep the maximum number of overlaps low; cf. e.g. the majority of the examples in [4]. Accordingly, we consider the following

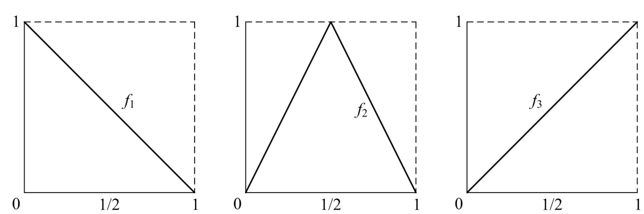

Fig. 2. A 3-overlapping family $\{f_1, f_2, f_3\}$.

definition. We say P is 2-*overlapping* if for all $x \in [0,1]$ and all triples of indices $i_1 \neq i_2 \neq i_3$ one has

$$\min\{f_{i_1}(x), f_{i_2}(x), f_{i_3}(x)\} = 0\ . \tag{2}$$

The Ruspini and the 2-overlapping conditions (1–2) apply to a family of fuzzy sets. In the literature, several properties applicable to a single fuzzy set have been considered too. One of these we are assuming throughout, as stated at the beginning; namely, continuity. Further, a fuzzy set $f\colon [0,1] \to [0,1]$ is *normal* if there exist $x \in [0,1]$ such that $f(x) = 1$. If, moreover, $f(y) \neq 1$ for all $y \in [0,1]$ with $y \neq x$, we say that f is *strongly normal*. The last property we wish to consider is convexity. Classically, $f\colon [0,1] \to [0,1]$

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[1]A deeper analysis of the logical notions involved here is not the subject of this paper. The interested reader may consult [2], [3] for the case of Gödel logic.

is *convex* if for all $x, y, \lambda \in [0, 1]$, with $x \neq y$,

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y). \quad (3)$$

Following [5, p. 25], it is common to consider a weaker form of convexity. The function f is *min-convex*[2] if for all $x, y, \lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \geq \min(f(x), f(y)), \quad (4)$$

and it is *strictly min-convex* if

$$f(\lambda x + (1 - \lambda)y) > \min(f(x), f(y)). \quad (5)$$

Finally, we shall need a localised version of convexity. Let us call $S_f = \{x \in [0, 1] \mid f(x) > 0\}$ the *support* of f. We say f is *convex on its support* if (3) holds for each $x, y \in [0, 1]$ such that $[x, y] \subseteq S_f$. We define the notions of (*strict*) *min-convexity of f on its support* in the same manner, *mutatis mutandis*.

Instead of asking that P (or its members) satisfy a given general property such as the ones above, we can decide to restrict the choice of fuzzy sets to a prototypical class of functions. So, for example, a fuzzy system might use only sigmoid, or triangular, or trapezoidal functions only. Concerning the triangular case, it is common to require that the various fuzzy sets fit together nicely, as in the following definition that is central to our paper.

Definition 1.1: A finite family $P = \{f_1, \ldots, f_n\}$ of continuous fuzzy sets is a *pseudo-triangular basis* if there exist $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ such that (up to a permutation of the indices) for each $i = 1, \ldots, n - 1$

a) $f_i(t_i) = 1$, $f_i(t_{i+1}) = 0$,
b) $f_j(x) = 0$, for $x \in [t_i, t_{i+1}]$, $j \neq i, i + 1$,
c) $f_{i+1}(x) = 1 - f_i(x)$, for $x \in [t_i, t_{i+1}]$, and
d) f_i, f_{i+1} are bijective when restricted to $[t_i, t_{i+1}]$.

Further, P is a *triangular basis* if the following condition holds in place of *d*).

*d**) f_i, f_{i+1} are linear over $[t_i, t_{i+1}]$.

Remark 1: It is straightforward to show that a finite family $\{f_1, \ldots, f_n\}$ of continuous fuzzy sets is a triangular basis if and only if there exist $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ such that (up to a permutation of the indices) for each $i \in \underline{n}$,

i) $f_i(t_i) = 1$,
ii) $f_i(t_j) = 0$, for $j \neq i$, and
iii) f_i is linear on each interval $[t_k, t_{k+1}]$, $k = 1, \ldots, n-1$.

Our first main result shows that (pseudo-)triangular bases of continuous fuzzy sets admit a characterisation in terms of the general properties discussed above. More precisely, in Theorem 2.3 we shall prove that P is a triangular basis if and only if it is a 2-overlapping Ruspini partition, and each f_i is strongly normal, min-convex, and convex on its support. To achieve this result, we build on the fact (Lemma 2.2) that P is a pseudo-triangular basis if and only if it is a 2-overlapping Ruspini partition, and each f_i is strongly normal, min-convex, and strictly min-convex on its support.

To remark on the significance of Theorem 2.3, let us return to our example about the temperature of a physical system. Suppose we were to design a fuzzy control system that has as input the temperature of the physical system. We wish to gather evidence that — for the control problem at hand — the choice of the triangular basis $\{f_1, f_2, f_3\}$ in Figure 1 is appropriate (at least initially). *Prima facie*, it is far from obvious how to go about arguing that the truth-degrees of our three sentences about temperature ought to obey the (implicit) assumptions encoded by $\{f_1, f_2, f_3\}$. However, our result does provide a line of attack: for one can do with an argument supporting the (explicit) properties listed in Theorem 2.3(i). Whether such an argument is there to be found, of course, depends on the problem under scrutiny.

Our second main result provides an additional characterisation of pseudo-triangular bases of fuzzy sets. Here, rather than looking for notable properties of P, we regard it from a different perspective. Namely, we interpret P as the parametric description of a curve whose image Θ lies in the real n-dimensional unit cube $[0, 1]^n$. We then seek to relate the properties of P discussed above with geometric properties of Θ. As it will transpire, some properties of P (such as whether P is a Ruspini partition) admit a characterisation in terms of the (closed) subset $\Theta \subseteq [0, 1]^n$; others, by contrast, require additional information on how the range Θ is parametrised by P. The characterisation of pseudo-triangular bases of fuzzy sets in terms of the associated parametrised curve is proved in Corollary 3.1.

II. Characterisation of triangular bases

Lemma 2.1: A fuzzy set f is min-convex if and only if for any $0 \leq x < z < y \leq 1$ we have that

$$\text{if } f(z) < f(x) \text{ then } f(y) \leq f(z)\,.$$

Moreover, f is strictly min-convex if and only if for any $0 \leq x < z < y \leq 1$ we have that

$$\text{if } f(z) \leq f(x) \text{ then } f(y) < f(z)\,.$$

Proof: This is a straightforward verification. ∎

Lemma 2.2: The following are equivalent.

i) P is a 2-overlapping Ruspini partition and each f_i is strongly normal, min-convex, and strictly min-convex on its support.
ii) P is a pseudo-triangular basis.

Proof: In the proof *a*), *b*), *c*), and *d*) will refer to the conditions in Definition 1.1.

i) ⇒ *ii*). Since each f_i is strongly normal there exist $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{n-1} \leq t_n \leq 1$ such that (up to a permutation of the indices) for each $i \in \underline{n}$

$$f_i(t_i) = 1,\ f_i(x) \neq 1,\ \text{for each } x \in [0, 1],\ x \neq t_i\,. \quad (6)$$

Moreover, since P is Ruspini we have

$$f_i(t_i) = 1,\ f_i(t_j) = 0,\ \text{for } j \neq i\,. \quad (7)$$

[2] We adopt this terminology to avoid confusion with convexity proper, which we shall also use.

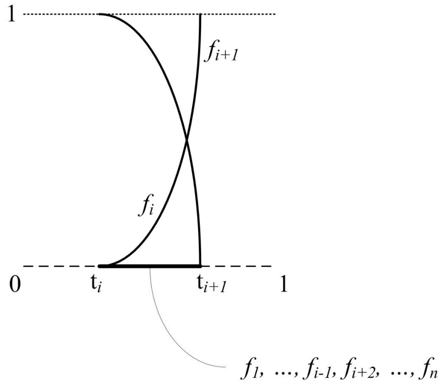

Fig. 3. A family of fuzzy sets satisfying *a*), *b*), *c*), and *d*).

Claim 1. $t_1 = 0, t_n = 1$.

Suppose $t_1 > 0$ (*absurdum hypothesis*). Then for each $i \in \underline{n}$, $f_i(0) < 1$. Since $\sum_{i=1}^{n} f_i(0) = 1$, and since P is 2-overlapping, there are exactly two indices $h > k \in \underline{n}$ such that $f_h(0), f_k(0) > 0$. Moreover, since $h > 1$, by (7) we have $f_h(t_1) = 0$ and $f_h(t_h) = 1$. By Lemma 2.1, we conclude that f_h is not min-convex, a contradiction. Thus, $t_1 = 0$. Analogously, one can prove $t_n = 1$, and the claim is settled.

By (7) and *Claim* 1, condition *a*) holds. In order to prove the other conditions in Definition 1.1 let us fix an interval $[t_i, t_i + 1]$, for some $i = 1, \ldots, n-1$.

To prove *b*), suppose by way of contradiction that there exists $j \neq i, i+1$ such that $f_j(x) > 0$ for some $x \in [t_i, t_{i+1}]$. Say $j < i$. Since, by (7), $x \neq t_i, t_{i+1}$, we have that, on $t_j < t_i < x$, f_j takes values $f_j(t_j) = 1$, $f_j(t_i) = 0$, $f_j(x) > 0$. . By Lemma 2.1, f_j is not min-convex, a contradiction. The argument for $j > i$ is analogous, and condition *b*) is proved.

From *b*) and the hypothesis that P is Ruspini, we immediately obtain *c*).

It remains to prove condition *d*). Recall that, by (7), $f_i(t_i) = f_{i+1}(t_{i+1}) = 1$ and $f_i(t_{i+1}) = f_{i+1}(t_i) = 0$. Moreover, since f_i and f_{i+1} are strongly normal, and P is Ruspini, using *b*) we have

$$0 < f_i(x), f_{i+1}(x) < 1\,, \text{ for all } x \in (t_i, t_{i+1})\,. \qquad (8)$$

Since f_i, f_{i+1} are continuous, by the intermediate value theorem they are surjective when restricted to $[t_i, t_{i+1}]$. Suppose now that there exist $y < z \in (t_i, t_{i+1})$ such that $f_i(y) = f_i(z)$ (*absurdum hypothesis*). Observe that, by (8), $[y, z]$ is contained in the support of f_i and f_{i+1}. Pick $w \in (y, z)$. If $f_i(w) \leq f_i(y)$, then, by Lemma 2.1, f_i is not strictly min-convex on its support, a contradiction. If $f_i(w) > f_i(y)$, then, by condition *c*),

$$f_{i+1}(w) = 1 - f_i(w) < 1 - f_i(y) = f_{i+1}(y) = f_{i+1}(z)\,.$$

Then, f_{i+1} is not strictly min-convex on its support, a contradiction. Therefore, f_i and f_{i+1} are injective, and *d*) holds.

ii) $\Rightarrow$ *i*). By condition *b*), P is 2-overlapping. Further, since *c*) holds, and P is 2-overlapping, P is Ruspini. Strong normality follows from *a*), *b*), and *d*) by direct inspection.

To prove min-convexity, suppose that for some $0 \leq x < y < z \leq 1$, and $i \in \underline{n}$, $f_i(x) > f_i(y)$. We note that, since f_n takes value 0 on $[0, t_{n-1}]$, and it is strictly increasing over $[t_{n-1}, 1]$, we have $i \neq n$. Since each f_i is increasing in $[0, t_i]$, we must have $y \in [t_i, 1]$. Furthermore, since each f_i is decreasing in $[t_i, 1]$, we have $f_i(z) \leq f_i(y)$. By Lemma 2.1, each f_i is min-convex.

The proof of the remaining strict min-convexity condition is analogous. ■

Theorem 2.3: The following are equivalent.

i) P is a 2-overlapping Ruspini partition, and each f_i is strongly normal, min-convex, and convex on its support.

ii) P is a triangular basis.

Proof: In the proof *a*), *b*), *c*), *d*), and *d**) will refer to the conditions in Definition 1.1.

i) $\Rightarrow$ *ii*). To prove *a*), *b*), and *c*) we proceed exactly as in the proof of Lemma 2.2, where such conditions are proved without using the strict min-convexity hypothesis.

To prove that P is a triangular basis, it remains to show that (up to a permutation of the indices)

L) f_i is linear over $[t_{i-1}, t_i]$, $i = 2, \ldots, n$,

R) f_i is linear over $[t_i, t_{i+1}]$, $i = 1, \ldots, n-1$.

Suppose, by way of contradiction, that f_i is not linear on $[t_i, t_{i+1}]$. By convexity, for every $\lambda \in [0, 1]$,

$$f_i(\lambda t_i + (1-\lambda)t_{i+1}) \geq \lambda f_i(t_i) + (1-\lambda) f_i(t_{i+1})\,.$$

If f_i is not linear, there exists $\lambda_0 \in [0, 1]$ such that

$$f_i(\lambda_0 t_i + (1-\lambda_0)t_{i+1}) > \lambda_0 f_i(t_i) + (1-\lambda_0) f_i(t_{i+1})\,. \quad (9)$$

By (9) and condition *c*), we have

$$1 - f_{i+1}(\lambda_0 x + (1-\lambda_0)y) > \lambda_0(1 - f_{i+1}(x)) + (1-\lambda_0)(1 - f_{i+1}(y))\,.$$

Therefore,

$$f_{i+1}(\lambda_0 x + (1-\lambda_0)y) < \lambda_0 f_{i+1}(x)(1-\lambda_0) f_{i+1}(y)\,,$$

and f_{i+1} is not convex on its support, a contradiction. This proves *R*). The case *L*) is analogous.

ii) $\Rightarrow$ *i*). Since *a*), *c*) and *d**) together imply *d*), by Lemma 2.2 P is a 2-overlapping Ruspini partition, and each f_i is strongly normal and min-convex. The proof of convexity is a straightforward verification. ■

III. The Curve Parametrised by a Pseudo-Triangular Basis

We define a continuous map

$$T : [0, 1] \to [0, 1]^n$$

associated with P by

$$t \mapsto (f_1(t), \ldots, f_n(t))\,.$$

Throughout, we write $\Theta = T([0,1])$ for the range of T.

Recall[3] that the *fundamental simplex* in $\mathbb{R}^n$, denoted by Δ_n, is the convex hull of the standard basis of $\mathbb{R}^n$; the latter is denoted $\{e_1, \ldots, e_n\}$. In symbols,

$$\Delta_n = \mathrm{Conv}\,\{e_1, \ldots, e_n\}\,.$$

A *face* of dimension k of Δ_n is a subset $\mathrm{Conv}\,\{e_{i_1}, \ldots, e_{i_{k+1}}\} \subseteq \Delta_n$, for $1 \leq i_1 < i_2 < \cdots < i_{k+1} \leq n$. A *vertex* is a 0-dimensional face. The 1-*skeleton* of Δ_n, written $\Delta_n^{(1)}$, is the collection of all faces of Δ_n having dimension not greater than 1.[4]

We say Θ is a *Hamiltonian path* if there is a permutation $\pi : \underline{n} \to \underline{n}$ such that

$$\Theta = \bigcup_{i=1}^{n-1} \mathrm{Conv}\,\{e_{\pi(i)}, e_{\pi(i+1)}\} \tag{10}$$

Corollary 3.1 (to Lemma 2.2): The following are equivalent.

i) P is a 2-overlapping Ruspini partition, and each f_i is strongly normal, min-convex, and strictly min-convex on its support.

ii) The map $T : [0,1] \to [0,1]^n$ is injective, and Θ is a Hamiltonian path on $\Delta_n^{(1)}$.

Proof: We first show the following.

Claim 1. P is a 2-overlapping Ruspini partition if and only if $\Theta \subseteq \Delta_n^{(1)}$.

$\Rightarrow$. Since P is Ruspini, $\Theta \subseteq \Delta_n$. Since P is 2-overlapping and $\Theta \subseteq \Delta_n$, we have $\Theta \subseteq \Delta_n^{(1)}$.

$\Leftarrow$. Since $\Theta \subseteq \Delta_n$, we have $\sum_{i=1}^{n} f_i(t) = 1$ for all $t \in [0,1]$. Thus, P is a Ruspini partition. Moreover, since $\Theta \subseteq \Delta_n^{(1)}$, $(f_1(x), \ldots, f_n(x))$ has at most 2 non-zero coordinates, for all $x \in [0,1]$. Thus, P is 2-overlapping.

i) $\Rightarrow$ *ii)*. By *Claim* 1, $\Theta \subseteq \Delta_n^{(1)}$. By Lemma 2.2, there exist $0 = t_1 < t_2 < \cdots < t_n = 1$ such that, up to a permutation of the indices,

$$T(t_i) = e_i\,, \text{ for all } x \in \underline{n}\,. \tag{11}$$

Moreover, for $x \in [0,1]$, $x \neq t_i$, and for every $x \in \underline{n}$,

$$T(x) \notin \Delta_n^{(0)}\,. \tag{12}$$

Let us fix an interval $[t_i, t_{i+1}]$, for some $i \in \{1, \ldots, n-1\}$. By Lemma 2.2, for all $x \in (t_i, t_{i+1})$, we have $0 < f_i(x), f_{i+1}(x) < 1$, and $f_j(x) = 0$ for $j \neq i, i+1$. Thus, $T((t_i, t_{i+1})) \subseteq \mathrm{Conv}\,(e_i, e_{i+1})$. Since each f_i is continuous, by the intermediate value theorem we have $\mathrm{Conv}\,(e_i, e_{i+1}) \subseteq T((t_i, t_{i+1}))$. Thus,

$$T([t_i, t_{i+1}]) = \mathrm{Conv}\,(e_i, e_{i+1})\,. \tag{13}$$

[3]For background on the few basic notions from piecewise linear geometry we use here, please see [6].

[4]Thus, $\Delta_n^{(1)}$ happens to be a graph.

By (11), (12), and (13), Θ is a Hamiltonian path. Further, the map T reaches each vertex $e_1, \ldots, e_n$ exactly once (in order).

To complete the proof we need to show that T is injective. Suppose T is not (*absurdum hypothesis*). Thus, there exist $x, y \in [0,1]$, with $x < y$ such that $T(x) = T(y)$. By (13), we have that $x, y \in (t_i, t_{i+1})$, for some i. Moreover, $f_i(t_i) = 1 > f_i(x) = f_i(y) > 0$. Observing that $[t_i, y]$ is contained in the support of f_i, by Lemma 2.1, f_i is not strictly min-convex on its support, a contradiction. Therefore T_i is injective, as was to be shown.

ii) $\Rightarrow$ *i)*. Since Θ is a Hamiltonian path on $\Delta_n^{(1)}$, we have $\Theta \subseteq \Delta_n^{(1)}$ and, by *Claim* 1, P is a 2-overlapping Ruspini partition. By the definition of Hamiltonian path, Θ contains all vertices of $\Delta_n^{(1)}$. Thus, each f_i is normal. Since, moreover, T is injective, each f_i is strongly normal. We obtain that, up to a permutation of the indices, there exist $t_1 < t_2 < \cdots < t_{n-1} < t_n$, such that $T(t_1) = e_1$, $T(t_2) = e_2$, $\ldots$, $T(t_{n-1}) = e_{n-1}$, $T(t_n) = e_n$. Moreover, for all $x \in [0,1]$, $x \neq t_i$, we have $T(x) \neq e_i$.

Claim 2. $t_1 = 0$, $t_n = 1$.

Suppose $t_1 > 0$ (*absurdum hypothesis*). By continuity, there exists $x \in (0, t_1)$ such that $0 < f_1(x) = a < 1$. Since P is a 2-overlapping Ruspini partition there exists i such that $f_i(x) = 1 - a$. Moreover, $f_j(x) = 0$, for $j \neq 1, i$. Since Θ is a Hamiltonian path we must have $i = 2$, for else Condition (10) does not hold. Thus, $T(x) = (a, 1-a, 0, \ldots, 0)$. By the intermediate value theorem, there exists $y \in (t_1, t_2)$ such that $f_1(y) = a$. Clearly, we have $f_2(y) = 1 - f_1(y) = 1 - a$. Thus, for $y \neq x$, we have $T(y) = (a, 1-a, 0, \ldots, 0) = T(x)$, again the hypothesis that T is injective. Therefore, $t_1 = 0$. Analogously, one can prove $t_n = 1$, and the Claim is settled.

It is now immediate to verify that conditions *a*), *b*), and *c*) in Definition 1.1 hold for the family P. To verify the last condition *d*) we just observe that each function in P is continuous and that T is injective. Thus, by the intermediate value theorem, f_i and f_{i+1} are bijective when restricted to $[t_i, t_{i+1}]$. Therefore, P is a pseudo-triangular basis. Using Lemma 2.2, the remaining of our statement is proved. ■

We illustrate with two examples, and a non-example.

Figure 4 show the range of the family $\{f_1, f_2, f_3\}$ depicted in Figure 1.

Next consider the family $\{f_1, f_2, f_3, f_4\}$ in Figure 5. It is easy to check that $\{f_1, f_2, f_3, f_4\}$ is a 2-overlapping Ruspini partition and that each f_i is strongly normal, min-convex, and strictly min-convex on its support. The associated range is depicted in Figure 6.

Finally, the non-example. Consider the non-Ruspini family $\{f_1, f_2, f_3\}$ depicted in Figure 7. One can see that the associated range, depicted in Figure 8, is not a Hamiltonian path.

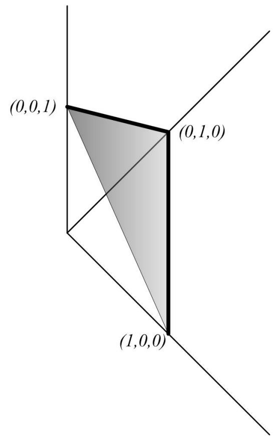

Fig. 4. Range parametrised by a (pseudo-)triangular basis with 3 functions.

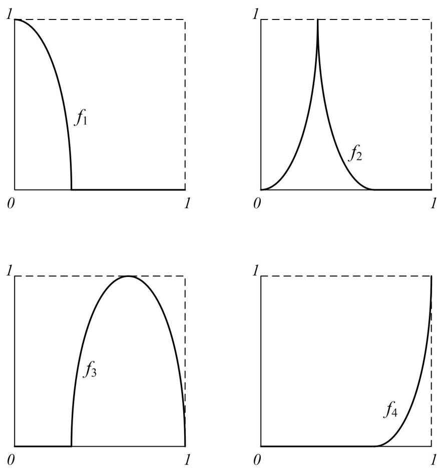

Fig. 5. A pseudo-triangular basis $\{f_1, f_2, f_3, f_4\}$.

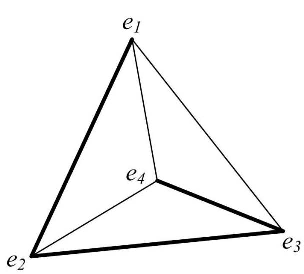

Fig. 6. Range parametrised by a pseudo-triangular basis with 4 functions.

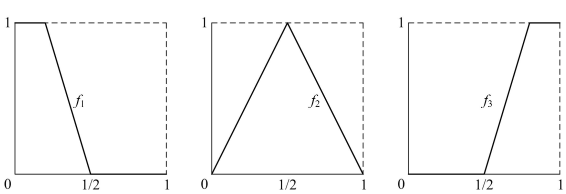

Fig. 7. A non-Ruspini family $\{f_1, f_2, f_3\}$.

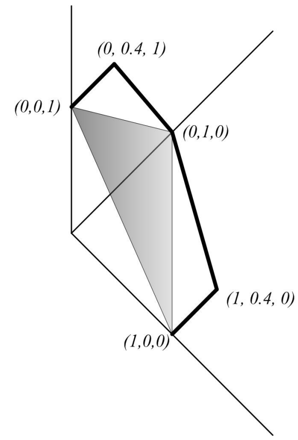

Fig. 8. Range parametrised by a non-Ruspini family $\{f_1, f_2, f_3\}$.

IV. Conclusion

Fuzzy sets featuring in applications are often required to satisfy specific conditions such as, e.g., convexity or normality. In this paper, we have shown that such conditions may be conceived as weaker forms of the requirement that the fuzzy sets have triangular shape. More precisely, our main result (Theorem 2.3) shows that triangular bases of fuzzy sets are precisely those families of continuous, strongly normal, min-convex fuzzy sets convex on their supports that form a 2-overlapping Ruspini partition. We have further shown (Corollary 3.1) that the more general notion of pseudo-triangular basis of fuzzy sets can be characterised by appropriate properties of the parametrised curve naturally associated with it.

In this paper, we focused on fuzzy sets whose domain is the real unit interval $[0, 1]$. In the presence of several physical observables, it may be necessary to deal with fuzzy sets defined over the real unit n-cube $[0, 1]^n$. A natural question is whether Theorem 2.3 admits a generalisation to higher dimensions. To make this question precise, one needs a notion of triangular basis over $[0, 1]^n$. If one is willing to restrict attention to (continuous) piecewise linear functions, a natural such notion is obtained as follows. (For background, please see [6]). Let Σ be a triangulation of $[0, 1]^n$. For each vertex v of Σ, consider the uniquely determined piecewise linear function h_v that takes value 1 at v, value 0 at any other vertex different from v, and is linear over each simplex of Σ. The collection $H_\Sigma = \{h_v \mid v \text{ is a vertex of } \Sigma\}$ is a *triangular basis* of fuzzy sets (over Σ). To the best of our knowledge, and in sharp contrast to the elementary nature of Theorem 2.3, all characterisations of such families of fuzzy sets make use of non-trivial topological conditions. For a result of this sort in the context of Łukasiewicz infinite-valued propositional logic and its algebraic counterpart – MV-algebras – interested readers are referred to [7, Theorem 4.5]. Further research may lead to simpler characterisations of triangular bases of fuzzy sets.

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