Propositional Gödel Logic and Delannoy Paths.

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***Abstract*— Gödel propositional logic is the logic of the minimum triangular norm, and can be axiomatized as propositional intuitionistic logic augmented by the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$. Its algebraic counterpart is the subvariety of Heyting algebras satisfying prelinearity, known as *Gödel algebras*. A *Delannoy path* is a lattice path in $\mathbb{Z}^2$ that only uses northward, eastward, and northeastward steps. We establish a representation theorem for free n-generated Gödel algebras in terms of the Boolean n-cube $\{0,1\}^n$, enriched by suitably generalized Delannoy paths.**

I. Introduction

Gödel (*infinite-valued propositional*) *logic* [1], [2], also known as *Gödel-Dummett logic*, can be semantically defined as the logic of the minimum triangular norm.

In more detail, first endow the real unit interval $[0,1]$ with the operations $\wedge$ and $\to$ defined by

$$x \wedge y = \min(x,y) \quad \text{and} \quad x \to y = \begin{cases} 1 & \text{if } x \leq y\,, \\ y & \text{otherwise}\,. \end{cases}$$

Next consider the set of well formed formulas defined in the usual manner from propositional variables X_i, $i = 1, 2, \ldots$, say, using the conjunction and implication connectives $\wedge$ and $\to$, along with the constant $\bot$. Then Gödel logic has as tautologies precisely the well formed formulas $\alpha(X_1, \ldots, X_n)$ that evaluate constantly to 1 under any $[0,1]$-assignment to the propositional variables X_i, when each connective is interpreted as the operation denoted by the same symbol, and the constant $\bot$ is interpreted as the real number 0.

Usual derived connectives are

- $\alpha \vee \beta := ((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)$
- $\neg\alpha := \alpha \to \bot$
- $\top := \neg\bot$

whose semantical counterparts in $[0,1]$ are

- $x \vee y = \max(x,y)$
- $\neg x = \begin{cases} 1 & \text{if } x = 0\,, \\ 0 & \text{otherwise.} \end{cases}$
- $\top = 1$.

Gödel logic can be axiomatized in Hilbert style as the intuitionistic propositional calculus augmented by the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$. For a completeness theorem with respect to the many-valued semantics above, see e.g. [2, 10.1.3].

In this paper we exhibit an unorthodox semantics for Gödel logic in terms of certain classical combinatorial objects, namely, *Delannoy paths* [3, p. 80]. Our main result is a representation theorem for free n-generated Gödel algebras based on such paths. In Section II we enrich the Boolean n-cube $\{0,1\}^n$ by (suitably generalized) Delannoy paths. We then introduce *Delannoy functions* as a generalization of Boolean functions. We observe that such Delannoy functions indeed form a Heyting algebra. In Section III, we clarify the tight connection between Delannoy paths in $\{0,1\}^n$ and $[0,1]$-assignments to n propositional variables in Gödel logic (Lemma 3.3). Finally, in Section IV, we prove that the Heyting algebra of Delannoy functions on n variables is the free Gödel algebra on n free generators. Our results should be compared with [4] where, among other things, the latter algebra is described in terms of another combinatorial construction.

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II. Delannoy paths and functions

According to the classical definition, a *Delannoy path* is a path in $\mathbb{Z}^2$ that only uses northward, eastward, and northeastward steps. We need a variant of this notion in order to work in Boolean n-cubes. Throughout, n is a nonnegative integer.

Definition 2.1: A *Delannoy step* in $\{0,1\}^n$ (*D-step* for short) is a pair (a,b) with $a = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ and $b = (y_1, y_2, \ldots, y_n) \in \{0,1\}^n$, $a \neq b$, such that $y_i \leq x_i$, for each $i \in \{1, \ldots, n\}$. We denote by S^a_b the D-step (a,b). A *Delannoy path* in $\{0,1\}^n$ is a sequence of D-steps in $\{0,1\}^n$ of the form $S^{a_0}_{a_1} S^{a_1}_{a_2} \cdots S^{a_{k-1}}_{a_k}$. We say that a_0 is the *root* of such a D-path. A path of k steps is a *k-path*. A 0-path is a point, and we shall simply denote it by its coordinates.

Figure 1 displays some D-paths in $\{0,1\}^3$.

Definition 2.2: Let $P = S^{a_0}_{a_1} S^{a_1}_{a_2} \cdots S^{a_{k-1}}_{a_k}$ be a D-path. A *subpath* of P is a D-path $Q = S^{a_0}_{a_1} S^{a_1}_{a_2} \cdots S^{a_{j-1}}_{a_j}$, $j \leq k$, consisting of the first j steps of P. We write $Q \sqsubseteq P$. We denote by $\Downarrow P$ the family of all subpaths of P.

Notice that the smallest subpath of a path P with root r is the 0-path with root r, that is, the point r. Also notice that all subpaths of P have root r.

We denote by $\mathcal{P}_n$ the set of all D-paths in $\{0,1\}^n$. Using S as a shorthand for S^1_0, we thus have

$$\mathcal{P}_1 = \{0, 1, S\}\,.$$

Note that the set of 0-paths in $\mathcal{P}_n$ is in bijection with the points of $\{0,1\}^n$.

Definition 2.3: A *Delannoy function* is a function $f : \mathcal{P}_n \to \mathcal{P}_1$ such that

$$f(\Downarrow P) = \Downarrow f(P)\,, \text{ for every } P \in \mathcal{P}_n\,.$$

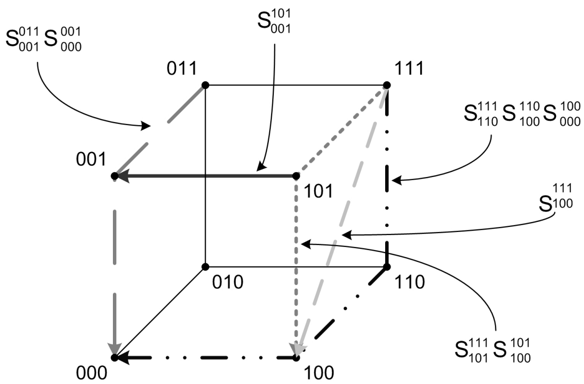

Fig. 1. Some D-paths in the 3-cube.

Restricting the domain of Delannoy functions to the set of all 0-paths we recover the classical case of Boolean functions $f : \{0,1\}^n \to \{0,1\}$.

Definition 2.4: Given a Delannoy function $f : \mathcal{P}_n \to \mathcal{P}_1$, the set $f^{-1}(1) = \{P \in \mathcal{P}_n : f(P) = 1\}$ is called the *1-set* of f.

Note that the 1-set $f^{-1}(1)$ is closed under the operation of taking subpaths, *i.e.* $\Downarrow f^{-1}(1) = f^{-1}(1)$.

Lemma 2.5: A Delannoy function $f : \mathcal{P}_n \to \mathcal{P}_1$ is uniquely determined by its 1-set.

Proof: Let P be a path in $\mathcal{P}_n$, $P \notin f^{-1}(1)$. Note that there does not exist a path $Q \in f^{-1}(1)$ such that $P \sqsubseteq Q$. We next consider two cases. Suppose there exists a path $Q \in f^{-1}(1)$ such that $Q \sqsubseteq P$. Then, $1 \in f(\Downarrow P)$ and, by the definition of Delannoy function, $1 \in \Downarrow f(P)$. Since $f(P) \neq 1$, the only other possibility is $f(P) = S$. If, on the other hand, there is no path $Q \in f^{-1}(1)$ such that $Q \sqsubseteq P$, then $1 \notin f(\Downarrow P) = \Downarrow f(P)$. Thus, $f(P) = S$ cannot hold, for else we would have $\Downarrow f(P) = \{1, S\}$. Hence, $f(P) = 0$. ∎

Consider now the set $\mathcal{D}_n$ of all Delannoy functions from $\mathcal{P}_n$ to $\mathcal{P}_1 = \{0, 1, S\}$. Let us endow $\mathcal{D}_n$ with binary operations $\wedge, \vee, \to$, defined pointwise as follows, for $P \in \mathcal{P}_n$, and $f, g \in \mathcal{D}_n$:

TABLE I

$f(P)$	$g(P)$	$(f \wedge g)(P)$	$(f \vee g)(P)$	$(f \to g)(P)$
0	0	0	0	1
0	S	0	S	1
0	1	0	1	1
S	0	0	S	0
S	S	S	S	1 if $M \sqsubseteq N$, S otherwise.
S	1	S	1	1
1	0	0	1	0
1	S	S	1	S
1	1	1	1	1

In the table above, the symbols M, N denote the $\sqsubseteq$-maximal subpaths of P such that $f(M) = g(N) = 1$. Such subpaths always exist, as one easily verifies using the definition of Delannoy function.

The following lemma, to be used in the sequel, is promptly established. We omit the proof for the sake of brevity.

Lemma 2.6: $\mathcal{D}_n$ is closed under the operations $\wedge, \vee, \to$. In particular, the constant functions c_0, c_1 that send every $P \in \mathcal{P}_n$ to the element $0 \in \mathcal{P}_1$ and $1 \in \mathcal{P}_1$, respectively, belong to $\mathcal{D}_n$. Moreover, $\langle \mathcal{D}_n, \wedge, \vee, \to, c_0, c_1 \rangle$ is a Heyting algebra.

To illustrate the Heyting algebra structure of Delannoy functions, we derive the Hasse diagram of $\mathcal{D}_1$. Let us display a function $f \in \mathcal{D}_1$ by listing the images of each element of the domain:

$$(f(0)\ f(1)\ f(S)).$$

It is easy to verify that there are exactly six Delannoy functions in $\mathcal{D}_1$, namely:

$$(0\ 0\ 0)\,,\ (0\ 1\ S)\,,\ (0\ 1\ 1)\,,\ (1\ 0\ 0)\,,\ (1\ 1\ S)\,,\ (1\ 1\ 1)\,.$$

Using Table I we can immediately compute the Hasse diagram of $\mathcal{D}_1$. To illustrate the relationship between Delannoy functions and Gödel logic, let us show how to associate with each function a formula in the single variable X. To the identity function $(0\ 1\ S)$ one associates the formula X. Next notice that the function $(1\ 0\ 0)$ can be obtained as $(0\ 1\ S) \to c_0$. Recalling that, as usual, negation is defined by $\neg\alpha = \alpha \to \bot$, we can write

$$(1\ 0\ 0) = \neg(0\ 1\ S).$$

Hence, one associates the formula $\neg X$ with the Delannoy function $(1\ 0\ 0)$. Similarly, observe that $(0\ 0\ 0) = (0\ 1\ S) \wedge (1\ 0\ 0)$ and that $(1\ 1\ S) = (0\ 1\ S) \vee (1\ 0\ 0)$. The remaining functions are promptly derived in a similar fashion. Table II summarizes the results.[1]

TABLE II

$(0\ 1\ S)$	X
$(1\ 0\ 0)$	$\neg X$
$(0\ 0\ 0)$	$X \wedge \neg X$
$(0\ 1\ 1)$	$\neg\neg X$
$(1\ 1\ S)$	$X \vee \neg X$
$(1\ 1\ 1)$	$\neg X \vee \neg\neg X$

In Figure 2 we show the Hasse diagram of $\mathcal{D}_1$ with nodes labelled by formulas as in Table II.

Note that the triplets describing Delannoy functions in $\mathcal{D}_1$ are ordered coordinatewise, according to the relations $0 \leq S \leq 1$.

Direct computation using Delannoy functions shows that $\mathcal{D}_2$ has 342 elements. (A recursive formula for the size of $\mathcal{D}_n$ can be found in [6], [7].)

[1] We remark that the analogous tables for more than one variable must mention the implication connective. Indeed, $\to$ is not definable from $\wedge, \vee, \neg, \bot, \top$ in Gödel logic [5].

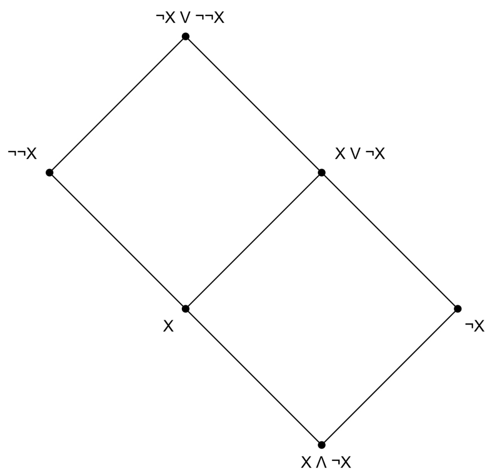

Fig. 2. The Heyting algebra $\mathcal{D}_1$.

III. Assignments vs. Delannoy paths

Let $(P, \leq)$ be a partially ordered set and $Q \subseteq P$. Recall that the *downset* of Q is

$$\downarrow Q = \{p \in P \mid p \leq q, \text{ for some } q \in Q\}.$$

We write $\downarrow p$ for $\downarrow \{p\}$. A partially ordered set $(P, \leq)$ is a *forest* if for all $q \in P$ the downset $\downarrow q$ is a chain (*i.e.*, a totally ordered set). A *tree* is a forest with a bottom element, called the *root* of the tree. A *subforest* of a forest P is the downset of some $Q \subseteq P$.

Unions and intersections of subforests are again subforests. Following [8], we define an implication between subforests by

$$F_1 \to F_2 = \{p \in P \mid \downarrow p \cap F_1 \subseteq \downarrow p \cap F_2\}, \qquad (1)$$

where F_1 and F_2 are subforests of $(P, \leq)$. It is easy to check that $F_1 \to F_2$ indeed is a subforest of $(P, \leq)$.

We now note that the set of Delannoy paths, $\mathcal{P}_n$, is partially ordered by the subpath relation, $\sqsubseteq$, and is in fact a forest. In Figure 3 we display $(\mathcal{P}_2, \sqsubseteq)$.

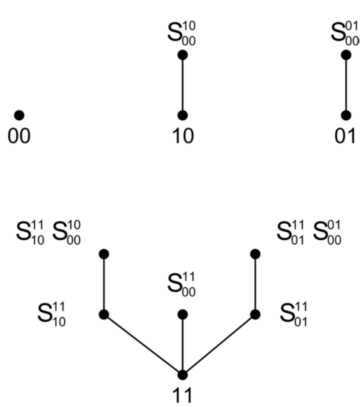

Fig. 3. The forest $\mathcal{P}_2$.

We next turn to assignments and their relationship with forests of Delannoy paths.

Let us consider well formed formulas over propositional variables X_i, $i = 1, 2, \ldots$. As is standard, by an *assignment* we mean a function μ from well formed formulas to $[0, 1] \subseteq \mathbb{R}$ such that, for any two such formulas α, β,

- $\mu(\alpha \wedge \beta) = \min\{\mu(\alpha), \mu(\beta)\}$.
- $\mu(\alpha \vee \beta) = \max\{\mu(\alpha), \mu(\beta)\}$.
- $\mu(\alpha \to \beta) = \begin{cases} 1, \text{ if } \mu(\alpha) \leq \mu(\beta), \\ \mu(\beta), \text{ otherwise.} \end{cases}$
- $\mu(\neg\alpha) = \begin{cases} 1, \text{ if } \mu(\alpha) = 0, \\ 0, \text{ otherwise.} \end{cases}$
- $\mu(\bot) = 0$
- $\mu(\top) = 1$

Definition 3.1: We say that two assignments μ and ν are *equivalent over the first* n *variables*, or *n-equivalent*, written $\mu \equiv_n \nu$, if and only if, for all $i, j \in \{1, \ldots, n\}$, the following conditions hold:

(a) $\mu(X_i) = 0 \iff \nu(X_i) = 0$,
(b) $\mu(X_i) = 1 \iff \nu(X_i) = 1$,
(c) $\mu(X_i) = \mu(X_j) \iff \nu(X_i) = \nu(X_j)$,
(d) $\mu(X_i) < \mu(X_j) \iff \nu(X_i) < \nu(X_j)$.

Clearly, $\equiv_n$ is an equivalence relation; let us write $\mathcal{F}_n$ for the (finite) set of equivalence classes of $\equiv_n$. The significance of this definition is that equivalent assignments cannot be told apart by the formulas they make true. Indeed, it is not difficult to show that if $\alpha(X_1, \ldots, X_n)$ is a well formed formula in Gödel logic, and μ, ν are two n-equivalent assignments, then

$$\mu(\alpha(X_1, \ldots, X_n)) = 1 \iff \nu(\alpha(X_1, \ldots, X_n)) = 1.$$

We further define a partial order on $\mathcal{F}_n$.

Definition 3.2: Let $[\mu]_{\equiv_n}$ and $[\nu]_{\equiv_n}$ be two equivalence classes of assignments. We define $[\mu]_{\equiv_n} \leq [\nu]_{\equiv_n}$ if and only if, for all $i, j \in \{1, \ldots, n\}$, the following conditions hold:

(i) $\mu(X_i) = 0 \iff \nu(X_i) = 0$,
(ii) $\mu(X_i) < 1 \implies \nu(X_i) < 1$,
(iii) $\mu(X_i) = \mu(X_j) < 1 \implies \nu(X_i) = \nu(X_j)$,
(iv) $\mu(X_i) < \mu(X_j) \implies \nu(X_i) < \nu(X_j)$.

One checks that $\leq$ in Definition 3.2 indeed is a partial order on $\mathcal{F}_n$. In fact, $(\mathcal{F}_n, \leq)$ and $(\mathcal{P}_n, \sqsubseteq)$ are order-isomorphic, as we now show.

Let us construct a function $F : \mathcal{P}_n \to \mathcal{F}_n$. Given a path $P = S^{a_0}_{a_1} S^{a_1}_{a_2} \cdots S^{a_{k-1}}_{a_k} \in \mathcal{P}_n$, let $a_0 = (y_1, \ldots, y_n), a_k = (z_1, \ldots, z_n) \in \{0, 1\}^n$. Then $F(P) = [\mu]_{\equiv_n}$ is defined as follows:
(*Variables set to* 0 *and* 1.) For $i \in \{1, \ldots, n\}$

$$\mu(X_i) = \begin{cases} 0 & \text{if } y_i = 0; \\ 1 & \text{if } z_i = 1; \\ 0 < \mu(X_i) < 1 & \text{if } y_i = 1 \text{ and } z_i = 0. \end{cases}$$

(*Order between variables.*) For each pair of indices $i, j \in \{1, \ldots, n\}$ such that $y_i = y_j = 1$ and $z_i = z_j = 0$ there is a unique D-step, say $S^{a_{r-1}}_{a_r}$, in which the i^{th} coordinate decreases, and a unique D-step, say $S^{a_{s-1}}_{a_s}$, in which the j^{th} coordinate decreases. Then:

- if $r < s$ then $\mu(X_i) < \mu(X_j)$;
- if $r = s$ then $\mu(X_i) = \mu(X_j)$;
- if $r > s$ then $\mu(X_i) > \mu(X_j)$.

For example, if we take $P = S^{111}_{110}S^{110}_{100} \in \mathcal{P}_3$, then $[\mu]_{\equiv_3} = F(P)$ is the class of those assignments such that

$$0 < \mu(X_3) < \mu(X_2) < \mu(X_1) = 1.$$

Lemma 3.3: The function $F : \mathcal{P}_n \to \mathcal{F}_n$ is a bijection. Moreover, for all $P \in \mathcal{P}_n$,

$$F(\Downarrow P) = \downarrow F(P). \tag{2}$$

In particular, $(\mathcal{F}_n, \leq)$ is a forest.

Proof: For each $[\mu]_{\equiv_n} \in \mathcal{F}_n$, we show how to construct a unique path $P \in \mathcal{P}_n$ such that $F(P) = [\mu]_{\equiv_n}$. Let π be a permutation of the indices $1, \ldots, n$ such that

$$0 \leq \mu(X_{\pi(1)}) \leq \mu(X_{\pi(2)}) \leq \cdots \leq \mu(X_{\pi(n)}) \leq 1.$$

Let $r_0 = 0$ if μ sets no variable to 0, and

$$r_0 = \max\{j \in \{1, \ldots, n\} \mid \mu(X_{\pi(j)}) = 0\}$$

otherwise. Similarly, let $s = n + 1$ if μ sets no variable to 1, and

$$s = \min\{j \in \{1, \ldots, n\} \mid \mu(X_{\pi(j)}) = 1\}$$

otherwise. To construct P, first note that the only possible choice for the root $a_0 = \{x^0_1, \ldots, x^0_n\}$ is

$$x^0_j = \begin{cases} 0 & \text{if } r_0 \geq 1 \text{ and } j \in \{\pi(1), \ldots, \pi(r_0)\}; \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, if $s = r_0 + 1$, then P is the 0-path a_0. Otherwise, P must have at least one step, say $S^{a_0}_{a_1}$ with $a_1 = \{x^1_1, \ldots, x^1_n\}$. Upon setting

$$r_1 = \max\{j \in \{r_0 + 1, \ldots, s - 1\} | \mu(X_{\pi(j)}) = \mu(X_{\pi(r_0+1)})\},$$

direct inspection shows that a_1 is uniquely determined as:

$$x^1_j = \begin{cases} 0 & \text{if } j \in \{\pi(1), \ldots, \pi(r_1)\}; \\ 1 & \text{if } j \in \{\pi(r_1 + 1), \ldots, \pi(n)\}. \end{cases}$$

If now $s = r_1 + 1$, then P is the 1-path $S^{a_0}_{a_1}$. Otherwise, iterate the construction.

It remains to show (2). But since F is a bijection, to prove (2) suffices to show that F is order preserving. To this end let P be a k-path of $\mathcal{P}_n$ and Q an h-subpath of P. Let $F(P) = [\mu]_{\equiv_n}$ and $F(Q) = [\nu]_{\equiv_n}$. We only need to show $[\nu]_{\equiv_n} \leq [\mu]_{\equiv_n}$, according to Definition 3.2. We immediately notice by the first clause in the definition of F that

$$\mu(X_j) = 0 \Longleftrightarrow \nu(X_j) = 0,$$

for each $j = 1, \ldots, n$ and (i) in Definition 3.2 is verified. Similarly, one checks that (ii) in Definition 3.2 is satisfied. Moreover, if $\nu(X_r) = \nu(X_s) < 1$, for $r, s \in \{1, \ldots, h\}$, then there is a step in Q in which the r^{th} and the s^{th} coordinates decrease together. The same step belongs to P, hence $\mu(X_r) = \mu(X_s) < 1$, and Condition (iii) is verified. Finally, if $\nu(X_r) < \nu(X_s)$ then there is a step of Q in which the r^{th} coordinate decreases while the s^{th} keeps its value 1. Since this step also is in P, $\mu(X_r) < \mu(X_s)$ and the last condition in Definition 3.2 is met. ■

In Figure 4 we display $(\mathcal{F}_2, \leq)$; compare with Figure 3.

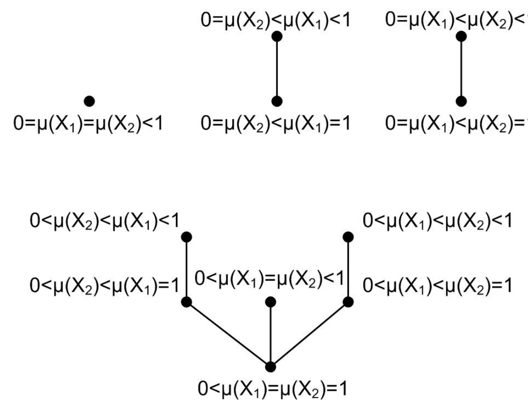

Fig. 4. The forest $\mathcal{F}_2$.

IV. MAIN RESULT

Let us denote by $\mathrm{Sub}(\mathcal{F}_n)$ the set of all subforests of $\mathcal{F}_n$. Recall from Section II that $\mathrm{Sub}(\mathcal{F}_n)$ comes equipped with operations $\cap, \cup, \to$. Moreover, the lattice $\langle \mathrm{Sub}(\mathcal{F}_n), \cap, \cup \rangle$ has $\mathcal{F}_n$ and $\emptyset$ as top and bottom, respectively.

Certain subforests of $\mathcal{F}_n$ correspond to propositional variables.

Definition 4.1: For each $i = 1, \ldots, n$, we define

$$\chi_i = \{[\mu]_{\equiv_n} \mid \mu(X_i) = 1\}$$

to be the i^{th} *generating subforest* of $\mathcal{F}_n$.

We prepare a lemma.

Lemma 4.2: $\langle \mathrm{Sub}(\mathcal{F}_n), \cap, \cup, \to, \emptyset, \mathcal{F}_n \rangle$ is a Gödel algebra freely generated by the generating subforests.

Proof: The proof is a straightforward translation of [7, Remark 2 and Proposition 2.4] in the language of assignments (Section III). ■

The Delannoy analog of generating subforests are projection functions:

Definition 4.3: Let $P = S^{a_0}_{a_1}S^{a_1}_{a_2} \cdots S^{a_{k-1}}_{a_k} \in \mathcal{P}_n$. Let $a_0 = (x_1, x_2, \ldots, x_n)$ and $a_k = (y_1, y_2, \ldots, y_n)$. We define the *projection functions* $\pi_i : \mathcal{P}_n \to \mathcal{P}_1,\ i \in \{1, \ldots, n\}$ by

$$\pi_i(P) = \begin{cases} 0 & \text{if } x_i = 0; \\ 1 & \text{if } y_i = 1; \\ S & \text{if } x_i = 1 \text{ and } y_i = 0. \end{cases}$$

One immediately checks that each projection function is a Delannoy function.

We are now ready to prove our main result.

Theorem 4.4: $\langle \mathcal{D}_n, \wedge, \vee, \to, c_0, c_1 \rangle$ is a Gödel algebra freely generated by the projection functions $\pi_1, \ldots, \pi_n$.

Proof: By Lemma 2.6, $\langle \mathcal{D}_n, \wedge, \vee, \to, c_0, c_1 \rangle$ is a Heyting algebra.

Let us define a function $\varphi : \mathcal{D}_n \to \mathrm{Sub}(\mathcal{F}_n)$ as follows. Given $f \in \mathcal{D}_n$, let $U \subseteq \mathcal{P}_n$ be the 1-set of f, that is, $U = \{P \in \mathcal{P}_n : f(P) = 1\}$. Let

$$f \in \mathcal{D}_n \stackrel{\varphi}{\mapsto} F(U) \in \mathrm{Sub}(\mathcal{F}_n),$$

where $F : \mathcal{P}_n \to \mathcal{F}_n$ is as in Lemma 3.3. In plain words, $\varphi(f)$ is the family of those classes $[\mu]_{\equiv_n}$ which are images under F of a path $P \in U$. By Lemma 3.3, it follows that $F(U) \in \mathrm{Sub}(\mathcal{F}_n)$, hence φ is well defined. Moreover, by the same Lemma, F is a bijection. Since f is uniquely determined by its 1-set (Lemma 2.5), φ is a bijection too.

We need to show that φ preserves operations. That $\varphi(c_0) = \emptyset$ and $\varphi(c_1) = \mathcal{F}_n$ is immediately seen. Let $P \in \mathcal{P}_n$ and $F(P) = [\mu]_{\equiv_n} \in \mathcal{F}_n$. Note that the intersection of the 1-sets of f and g is precisely the 1-set of $f \wedge g$. Therefore,

$$[\mu]_{\equiv_n} \in \varphi(f \wedge g) \iff [\mu]_{\equiv_n} \in \varphi(f) \text{ and } [\mu]_{\equiv_n} \in \varphi(g) \iff$$
$$\iff [\mu]_{\equiv_n} \in \varphi(f) \cap \varphi(g),$$

and φ preserves $\wedge$. Similarly, φ preserves $\vee$.

Concerning implication, we recall from (1) that $\varphi(f) \to \varphi(g)$ is the set of all classes $[\mu]_{\equiv_n} \in \mathcal{F}_n$ such that $A \subseteq B$, where

$$A = \{[\lambda]_{\equiv_n} \in \varphi(f) \mid [\lambda]_{\equiv_n} \leq [\mu]_{\equiv_n}\},$$

and

$$B = \{[\lambda]_{\equiv_n} \in \varphi(g) \mid [\lambda]_{\equiv_n} \leq [\mu]_{\equiv_n}\}.$$

As before, consider $P \in \mathcal{P}_n$ with $F(P) = [\mu]_{\equiv_n}$. The proof requires a case-analysis to cover each row in Table I.

Suppose $f(P) = g(P) = S$ (row 5). In this case, $[\mu]_{\equiv_n} \notin \varphi(f)$, and $[\mu]_{\equiv_n} \notin \varphi(g)$. Let M, N be the $\sqsubseteq$-maximal subpaths of P such that $f(M) = g(N) = 1$. If $M \sqsubseteq N$, then $[\mu]_{\equiv_n} \in \varphi(f \to g)$. Moreover, $M \sqsubseteq N$ implies $A \subseteq B$, and $[\mu]_{\equiv_n} \in \varphi(f) \to \varphi(g)$. On the other hand, if $N \sqsubset M$, then $[\mu]_{\equiv_n} \notin \varphi(f \to g)$. In this case, $F(N) \in B$, but $F(N) \notin A$. Thus, $A \nsubseteq B$, and $[\mu]_{\equiv_n} \notin \varphi(f) \to \varphi(g)$. In each case, implication is preserved.

Next suppose $f(P) = 0$ (rows $1-3$). Then $[\mu]_{\equiv_n} \notin \varphi(f)$. Furthermore, $(f \to g)(P) = 1$ and then $[\mu]_{\equiv_n} \in \varphi(f \to g)$. By Definition 2.3, $\Downarrow f(P) = \{0\} = f(\Downarrow P)$. This means that, for all paths $Q \sqsubseteq P$, $f(Q) = 0$, hence $F(Q) \notin \varphi(f)$. By Lemma 3.3, $Q \sqsubseteq P$ if and only if $F(Q) \leq F(P)$. Thus, A is empty, $A \subseteq B$, and $[\mu]_{\equiv_n} \in \varphi(f) \to \varphi(g)$. The remaining cases are dealt with in a similar fashion.

We have proved that $\langle \mathcal{D}_n, \wedge, \vee, \to, c_0, c_1 \rangle$ is isomorphic to $\langle \mathrm{Sub}(\mathcal{F}_n), \cap, \cup, \to, \emptyset, \mathcal{F}_n \rangle$ via φ. It remains to show that φ carries projection functions to generating subforests (see Figure 5 for an example).

Let therefore $\pi_i : \mathcal{P}_n \to \mathcal{P}_1$ be a projection function, and let $P = S^{a_0}_{a_1} S^{a_1}_{a_2} \cdots S^{a_{k-1}}_{a_k} \in \mathcal{P}_n$, $a_k = (y_1, y_2, \ldots, y_n)$. Then, P is in the 1-set of π_i if and only if $y_i = 1$. In this case, $[\mu]_{\equiv_n} = F(P)$ is such that $\mu(X_i) = 1$. Thus, $[\mu]_{\equiv_n}$ is in the generating subforest $\chi_i \subseteq \mathcal{F}_n$. Hence, φ maps π_i to $\chi_i \in \mathrm{Sub}(\mathcal{F}_n)$, and the proof is complete. ■

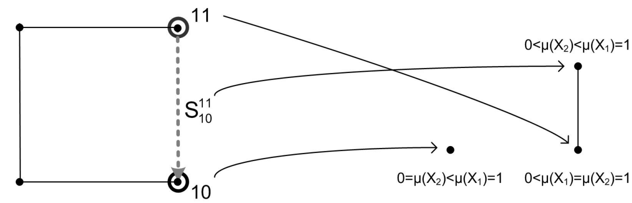

Fig. 5. $\chi_1 = \varphi(\pi_1)$.

V. Conclusions

In this paper, we have introduced a semantics for Gödel logic based on the combinatorial notion of Delannoy paths. As an example of further work along these lines, we mention the computation of coproducts of Gödel algebras exploiting the geometry of Delannoy paths, as an alternative to the ordered partitions used in [7].

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