The logical content of triangular bases of fuzzy sets in Łukasiewicz infinite-valued logic

Pietro Codara[a,**], Ottavio M. D'Antona[a], Vincenzo Marra[b,*]

[a] *Dipartimento di Informatica, Università degli Studi di Milano, via Comelico 39, I-20135 Milano, Italy*
[b] *Dipartimento di Matematica "Federigo Enriques", Università degli Studi di Milano, via Saldini 50, I-20133 Milano, Italy*

Abstract

Continuing to pursue a research direction that we already explored in connection with Gödel-Dummett logic and Ruspini partitions, we show here that Łukasiewicz logic is able to express the notion of *pseudo-triangular basis* of fuzzy sets, a mild weakening of the standard notion of triangular basis. *En route* to our main result we obtain an elementary, logic-independent characterisation of triangular bases of fuzzy sets.

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*Corresponding author.
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Email addresses: codara@di.unimi.it (Pietro Codara), dantona@di.unimi.it (Ottavio M. D'Antona), vincenzo.marra@unimi.it (Vincenzo Marra)

1. Prologue.

In this paper, by a *fuzzy set* we always mean a function $f\colon [0,1] \to [0,1]$, with $[0,1] \subseteq \mathbb{R}$, the real unit interval. Throughout, we fix an integer $n > 0$, and a finite, non-empty family

$$P = \{f_1, \ldots, f_n\}$$

of fuzzy sets. We further always assume that each $f_i \in P$ is a continuous function with respect to the usual (Euclidean) topology of $[0,1]$.

We address here the general question, what is the logical content of the family of fuzzy sets P. By way of motivation, let us think of the real unit interval $[0,1]$ as the normalised range of values of a physical observable, say temperature. Then each $f_i \in P$ can be viewed as a means of assigning a *truth value* to a proposition about temperature in some many-valued logic $\mathscr{L}$. Had one no information at all about such propositions, one would be led to identify them with propositional variables X_i, subject only to the axioms of $\mathscr{L}$. Intuitively, however, *the set P does encode information about $X_1, \ldots, X_n$*. For example, consider $P = \{f_1, f_2, f_3\}$ as in Fig. 1, and say f_1, f_2, and f_3 provide truth values for the propositions $X_1 =$ "The temperature is low", $X_2 =$ "The temperature is medium", and $X_3 =$ "The temperature is high", respectively. The shape of the functions in P intuitively tells us that it is never the case that the observed temperature is both low and high. More generally, at an intuitive level it is clear that P *encodes a body B of knowledge about the specific application domain* (here, about temperature). How can we make this intuition precise?

If $\mathscr{L}$ has a conjunction $\wedge$ interpreted by minimum, the proposition $X_1 \wedge X_3$ has 0 as its only possible truth value, *i.e.* it is a contradiction. The chosen set P then leads us to add *extra-logical axioms* to $\mathscr{L}$—*e.g.* $\neg(X_1 \wedge X_3)$, where $\neg$ is a negation connective—in an attempt to express the fact that one cannot observe both a high and a low temperature at the same time. More generally, we see that P *implicitly encodes a theory Θ_P over the pure logic $\mathscr{L}$*—a *theory* being a family of formulæ required to hold, thought of as extra-logical axioms. Crucially, the theory Θ_P is determined independently of the specifics of the available connectives. Set

$$\Theta_P = \{\varphi(X_1, \ldots, X_n) \mid \varphi(f_1(x), \ldots, f_n(x)) = 1 \text{ for all } x \in [0,1]\}\,. \qquad (*)$$

(Here, φ is a formula of $\mathscr{L}$ over the variables $X_1, \ldots, X_n$, and $\varphi(r_1, \ldots, r_n)$ denotes the evaluation of φ at $(r_1, \ldots, r_n) \in [0,1]^n$.) Under the sole assumption that $\mathscr{L}$ has a sound $[0,1]$-valued semantics, it is easy to show that Θ_P as in (*) is a (deductively closed) theory over $\mathscr{L}$; see Lemma 5.1 below.

In traditional terminology, the axioms of the logic $\mathscr{L}$ (along with their deductive consequences) are to be thought of as *analytic truths*, which hold true by virtue of their form alone, independently of the circumstances. Analytic truths are the subject matter of logic proper; however, by their very nature, they carry no information about "the world": whichever analytic truth one utters about temperature, one can equally well utter about, say, pressure. By contrast, the additional formulæ, or extra-logical axioms, that feature in a theory are to be thought of as *synthetic truths*—assertions that are truthful only within a specific domain of application, by virtue of properties of that domain. So, for example, in dealing with a certain (ideal) gas one may wish to assert as a *physical* (hence extra-logical) *truth* that the product of the volume and the pressure is constant at constant temperature. But there is of course no way of deducing such a statement from the axioms of classical logic: a world in which this specific law fails is conceivable, *i.e.* is logically consistent, and hence whatever the truth expressed by the law, it is a factual, or contingent, or synthetic truth. The completeness theorem then tells us that the statement in question is not formally derivable from the axioms of classical logic, because it has a counter-model, namely, the possible world wherein it fails. In this precise sense, *logic can teach us nothing* (factual): good grades in logic won't help with your physics class.

In light of the foregoing, we now see how to relate the two statements:

(S1) P determines a theory Θ_P over $\mathscr{L}$, and

(S2) P encodes a body B of knowledge about the specific application domain.

Indeed, Θ_P *is none other than a verbalisation of* B: an exposition of B in formulæ, so to speak. But while (S1) does provide the desired clarification of the intuition (S2), Θ_P may end up being a mere approximation to B; after all, the linguistic resources offered by $\mathscr{L}$ are limited. The differential relationship between acceleration and velocity, for example, is hardly exactly expressible within most formal system that go under the name of "logics".

In this paper we are concerned with one instance of the general problem of making explicit the extra-logical information implicitly encoded by P. In a previous paper [9] (see also [8]), we addressed and solved this problem in case the background logic $\mathscr{L}$ is *Gödel-Dummett* (*infinite-valued propositional*) *logic* [13, Chapter 4], and P is assumed to be a *Ruspini partition*, *i.e.* such that $\sum_{i=1}^{n} f_i(x) = 1$ for each $x \in [0, 1]$. There, we proved that Gödel-Dummett logic can only capture the semantical notion of Ruspini partition up to an equivalence relation that we determined exactly. Here, we address the problem of identifying the synthetic, factual content of *triangular bases of fuzzy sets*, a notion strictly stronger than Ruspini partitions. Such triangular bases commonly occur in applications. The set $P = \{f_1, f_2, f_3\}$ provides an example; please see Definition 2.2 below for details. Throughout the paper, we will take $\mathscr{L}$ to be *Łukasiewicz* (*infinite-valued propositional*) *logic* [5]; background is provided in Section 4. It is well known that Łukasiewicz logic is able to express addition of real numbers exactly, and so does axiomatise Ruspini partitions exactly. We shall prove in Theorem II of Section 6 the stronger result that Łukasiewicz logic axiomatises the notion of triangular bases of fuzzy sets almost exactly; specifically, the logic axiomatises *pseudo-triangular bases* (see Definition 2.2), a mild weakening of triangular bases. It will transpire that the reason why the latter cannot be characterised exactly is that the logic does not express (*affine*) *linearity*, though, as mentioned, it does express addition. Besides a fair amount of standard machinery in Łukasiewicz logic, the proof of Theorem II uses Theorem

I, which we prove in Section 3. Here we characterise pseudo-triangular bases of fuzzy sets by elementary properties of the set of functions P which strengthen the Ruspini condition. In the final Section 7 we discuss further research, and connections with previous work on the algebraic semantics of Łukasiewicz logic.

2. Properties of fuzzy sets.

Fuzzy sets are often required to satisfy additional conditions that are deemed useful for the specific application under consideration. Here is a popular one that we already mentioned in the Prologue, and is usually traced back[1] to [22, p. 28]. We say P is a *Ruspini partition* if for all $x \in [0,1]$

$$\sum_{i=1}^{n} f_i(x) = 1\,. \tag{1}$$

We further say that P is 2-*overlapping* if for all $x \in [0,1]$ and all triples of indices $i_1 \neq i_2 \neq i_3$ one has

$$\min\{f_{i_1}(x), f_{i_2}(x), f_{i_3}(x)\} = 0\ . \tag{2}$$

Figure 1 shows a family of fuzzy sets which is both Ruspini and 2-overlapping.

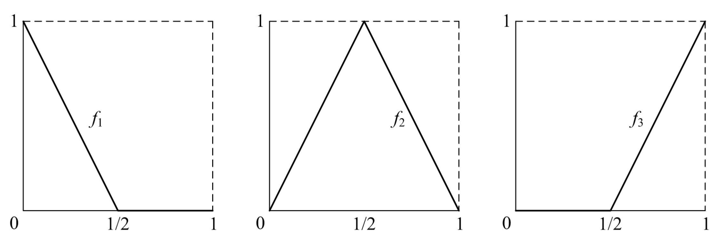
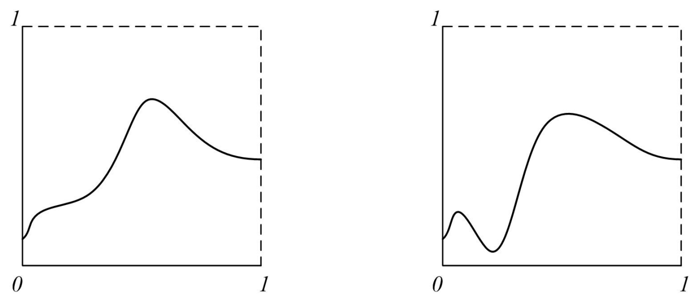


Figure 1: A Ruspini and 2-overlapping family of fuzzy sets.

The Ruspini and the 2-overlapping conditions (1–2) apply to a family of fuzzy sets. In the literature, several properties applicable to a single fuzzy set have been considered too. One of these we are assuming throughout, as stated at the beginning; namely, continuity. Further, a fuzzy set $f\colon [0,1] \to [0,1]$ is *normal* if there exist $x \in [0,1]$ such that $f(x) = 1$. If, moreover, $f(y) \neq 1$ for all $y \in [0,1]$ with $y \neq x$, we say that f is *strongly normal*. The fuzzy sets f_1, f_2, and f_3 depicted in Figure 1 are strongly normal. The last property we wish to consider is convexity. Following [11, p. 25], it is common to consider a

[1]Let us mention in passing that Ruspini partitions have been long studied in general topology, where they are known as (*finite*) *partitions of unity*; see *e.g.* the survey [12], and references therein.



(a) A strictly min-convex fuzzy set.

(b) A non-min-convex fuzzy set.

Figure 2: Min-convexity.

weaker form of convexity than the classical one. The fuzzy set $f\colon [0,1] \to [0,1]$ is *min-convex*[2] if for all $x, y, \lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \geq \min(f(x), f(y)), \tag{3}$$

and it is *strictly min-convex* if

$$f(\lambda x + (1-\lambda)y) > \min(f(x), f(y)). \tag{4}$$

We shall make crucial use of a localised version of min-convexity in our results. Let us call $S_f = \{x \in [0,1] \mid f(x) > 0\}$ the *support* of f. We say f is *min-convex on its support* if (3) holds for each $x, y \in [0,1]$ such that $[x,y] \subseteq S_f$. We define the notion of *strict min-convexity of f on its support* in the same manner, *mutatis mutandis*.

A min-convex fuzzy set is shown in Figure 2(a); a non-min-convex fuzzy set is shown in Figure 2(b).

Lemma 2.1. *A fuzzy set $f\colon [0,1] \to [0,1]$ is min-convex if, and only if, for any $0 \leq x < z < y \leq 1$ we have that*

$$\textit{if } f(z) < f(x) \textit{ then } f(y) \leq f(z)\,.$$

Moreover, f is strictly min-convex if, and only if, for any $0 \leq x < z < y \leq 1$ we have that

$$\textit{if } f(z) \leq f(x) \textit{ then } f(y) < f(z)\,.$$

Proof. This is a straightforward verification. ∎

There is one last property of fuzzy sets that we consider in this paper:

Definition 2.1. A finite family $P = \{f_1, \dots, f_n\}$ of fuzzy sets is *separating* if for all $x, y \in [0,1]$, with $x \neq y$, $\{f_1(x), \dots, f_n(x)\} \neq \{f_1(y), \dots, f_n(y)\}$.

[2] We adopt this terminology to avoid confusion with convexity proper.

It may be remarked that many families of fuzzy sets that have been investigated in the literature, or have been used in implementations, indeed are separating — the set in Figure 1 being a typical instance. We shall see in due course that the property is a crucial feature of such examples, *cf.* Theorem II below.

Instead of asking that P (or its members) satisfy a given general property such as the ones above, we can decide to restrict the choice of fuzzy sets to a prototypical class of functions. So, for example, a fuzzy system might use sigmoid, or triangular, or trapezoidal functions only. In the case of triangular functions, moreover, it is common to require that the various fuzzy sets fit together nicely, as in the following definition that is central to our paper.

Definition 2.2. A finite family $P = \{f_1, \ldots, f_n\}$ of continuous fuzzy sets is a *pseudo-triangular basis* if there exist $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ such that (up to a permutation of the indices) for each $i = 1, \ldots, n-1$

a) $f_i(t_i) = 1$, $f_i(t_{i+1}) = 0$,
b) $f_j(x) = 0$, for $x \in [t_i, t_{i+1}]$, $j \neq i, i+1$,
c) $f_{i+1}(x) = 1 - f_i(x)$, for $x \in [t_i, t_{i+1}]$, and
d) f_i, f_{i+1} are bijective when restricted to $[t_i, t_{i+1}]$.

Further, P is a *triangular basis* if the following condition holds in place of *d*).

*d**) f_i, f_{i+1} are linear over $[t_i, t_{i+1}]$.

See Figure 3 for an example of a pseudo-triangular basis of fuzzy sets.

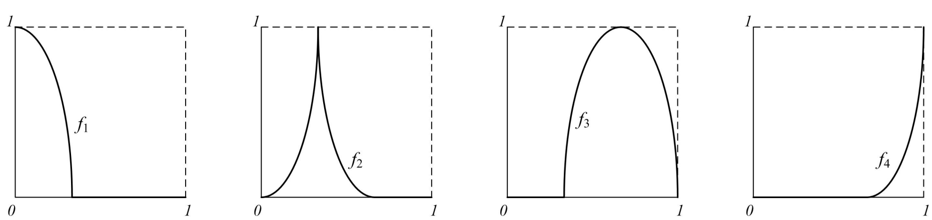


Figure 3: A pseudo-triangular basis.

Remark 1. It is straightforward to prove that a finite family $\{f_1, \ldots, f_n\}$ of continuous fuzzy sets is a triangular basis if, and only if, there exist real numbers $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ such that (up to a permutation of the indices) for each $i = 1, 2, \ldots, n$,

i) $f_i(t_i) = 1$,
ii) $f_i(t_j) = 0$, for $j \neq i$, and
iii) f_i is linear on each interval $[t_k, t_{k+1}]$, $k = 1, \ldots, n-1$.

The conditions in Definition 2.2 are somewhat more involved in order to capture instances that are not locally linear, as in Figure 3.

3. Characterisation of pseudo-triangular bases of fuzzy sets.

We define a continuous map

$$T_P\colon [0,1] \to [0,1]^n$$

associated with P by

$$t \mapsto (f_1(t), \ldots, f_n(t)) .$$

We write $\operatorname{ran} T_P = T_P([0,1])$ for the range of T_P.

Recall[3] that the *fundamental simplex* in $\mathbb{R}^n$, denoted by Δ_n, is the convex hull of the standard basis of $\mathbb{R}^n$; the latter is denoted $\{e_1, \ldots, e_n\}$. In symbols,

$$\Delta_n = \operatorname{Conv}\{e_1, \ldots, e_n\} .$$

A *face* of dimension k of Δ_n is a subset $\operatorname{Conv}\{e_{i_1}, \ldots, e_{i_{k+1}}\} \subseteq \Delta_n$, for $1 \leq i_1 < i_2 < \cdots < i_{k+1} \leq n$. A *vertex* is a 0-dimensional face. The 1-*skeleton* of Δ_n, written $\Delta_n^{(1)}$, is the collection of all faces of Δ_n having dimension not greater than 1.[4]

We say $\operatorname{ran} T_P$ is a *Hamiltonian path* if there is a permutation $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that

$$\operatorname{ran} T_P = \bigcup_{i=1}^{n-1} \operatorname{Conv}\{e_{\pi(i)}, e_{\pi(i+1)}\} \tag{5}$$

Theorem I. *The following are equivalent.*

i) *P is a pseudo-triangular basis.*
ii) *P is a 2-overlapping Ruspini partition and each $f_i \in P$ is strongly normal, min-convex, and strictly min-convex on its support.*
iii) *The map $T_P : [0,1] \to [0,1]^n$ is injective, and $\operatorname{ran} T_P$ is a Hamiltonian path on $\Delta_n^{(1)}$.*

Proof. Labels *a*), *b*), *c*), and *d*) in this proof refer to the items in Definition 2.2.

i) $\Rightarrow$ *iii*). By *b*) and *c*), we immediately obtain that

$$\operatorname{ran} T_P \subseteq \Delta_n^{(1)} . \tag{6}$$

By *a*), there exist $0 = t_1 < t_2 < \cdots < t_n = 1$ such that, up to a permutation of the indices,

$$T_P(t_i) = e_i\,, \text{ for each } i = 1, \ldots, n . \tag{7}$$

[3]For background on the few basic notions from piecewise linear geometry we use here, please see [21].

[4]Thus, $\Delta_n^{(1)}$ happens to be a graph.

Let us fix an interval $[t_i, t_{i+1}]$, for some $i \in \{1, \ldots, n-1\}$. By (6–7) and b), $T_P([t_i, t_{i+1}]) \subseteq \mathrm{Conv}\,\{e_i, e_{i+1}\}$. Again by (7), since T_P is continuous, using the intermediate value theorem we obtain

$$T_P([t_i, t_{i+1}]) = \mathrm{Conv}\,\{e_i, e_{i+1}\}\,. \tag{8}$$

Thus,

$$\mathrm{ran}\,T_P = \bigcup_{i=1}^{n-1} T([t_i, t_{i+1}]) = \bigcup_{i=1}^{n-1} \mathrm{Conv}\,\{e_i, e_{i+1}\}\,,$$

that is, $\mathrm{ran}\,T_P$ is a Hamiltonian path.

It remains to show that T_P is injective. If not (*absurdum hypothesis*), there exist $x, y \in [0,1]$, with $x < y$ such that $T_P(x) = T_P(y)$. Then (8) entails that $x, y \in [t_i, t_{i+1}]$, for some i. But the fact that $f_i(x) = f_i(y)$ and $f_{i+1}(x) = f_{i+1}(y)$, for $x \neq y$, contradicts d).

iii) $\Rightarrow$ *ii*). Since $\mathrm{ran}\,T_P \subseteq \Delta_n$, we have $\sum_{i=1}^{n} f_i(x) = 1$ for all $x \in [0,1]$, that is, P is a Ruspini partition. Since $\mathrm{ran}\,T_P \subseteq \Delta_n^{(1)}$, $(f_1(x), \ldots, f_n(x))$ has at most 2 non-zero coordinates, for each $x \in [0,1]$, that is, P is 2-overlapping. By the definition of Hamiltonian path, $\mathrm{ran}\,T_P$ contains all vertices of $\Delta_n^{(1)}$. Thus, each f_i is normal. Since, moreover, T_P is injective, each f_i is strongly normal.

Up to a permutation of the indices, there exist $t_1 < t_2 < \cdots < t_{n-1} < t_n$, such that $T_P(t_i) = e_i$, for each $i = 1, \ldots, n$. Moreover, by the intermediate value theorem we have $T_P([t_i, t_{i+1}]) \supseteq \mathrm{Conv}\,\{e_i, e_{i+1}\}$, $i = 1, \ldots, n-1$. But since T_P is injective it follows at once that

$$T_P([t_i, t_{i+1}]) = \mathrm{Conv}\,\{e_i, e_{i+1}\},\ i = 1, \ldots, n-1. \tag{9}$$

Now (9) implies, for each $i \in \{2, \ldots, n-1\}$,

$$f_i(x) < f_i(y)\,,\ \text{for } t_{i-1} \leq x < y \leq t_i\,; \tag{10}$$

$$f_i(x) > f_i(y)\,,\ \text{for } t_i \leq x < y \leq t_{i+1}\,; \tag{11}$$

$$f_i(x) = 0\,,\ \text{for } x \leq t_{i-1} \text{ or } x \geq t_{i+1}\,. \tag{12}$$

Indeed, (12) is immediate, and (10) follows from the injectivity of T_P: if $f_i(x) = f_i(y)$ then $f_{i-1}(x) = 1 - f_i(x) = 1 - f_i(y) = f_{i-1}(y)$, so that $T_P(x) = T_P(y)$, a contradiction. The proof of (11) is analogous. Similar arguments show that, for each $0 \leq x < y \leq 1$, $f_1(x) \geq f_1(y)$ and $f_n(x) \leq f_n(y)$.

We can now show that f_i is min-convex, for each $i = 1, \ldots, n$. By Lemma 2.1 it suffices to show that whenever $0 \leq x < y < z \leq 1$, and $f_i(x) > f_i(y)$, then $f_i(y) \geq f_i(z)$. The cases $i = 1$ and $i = n$ are trivial; assume $1 < i < n$. Since $f_i(x) > 0$, we have $t_{i-1} < x < t_{i+1}$ by (12). If $x \geq t_i$, by (11) and (12), $f_i(y) \geq f_i(z)$. If $x < t_i$, then, by (10), $y > t_i$. Using (11–12), we obtain $f_i(y) \geq f_i(z)$. In each case, if $f_i(x) > f_i(y)$, then $f_i(y) \geq f_i(z)$. A similar argument using (10–12) and Lemma 2.1 proves that each f_i is strictly min-convex on its support.

ii) ⇒ *i*). Since each f_i is strongly normal, and P is Ruspini, there exist $0 \leq t_1 < t_2 < \cdots < t_{n-1} < t_n \leq 1$ such that (up to a permutation of the indices) for each $i = 1, \ldots, n$ we have

$$f_i(t_i) = 1,\ f_i(t_j) = 0,\ \text{for } j \neq i\,. \tag{13}$$

Moreover, $t_1 = 0, t_n = 1$. For suppose $t_1 > 0$ (*absurdum hypothesis*). Then $f_i(0) < 1$ for each $i = 1, \ldots, n$. Since $\sum_{i=1}^n f_i(0) = 1$, and since P is 2-overlapping, there are exactly two indices $h > k \in \{1, \ldots, n\}$ such that $f_h(0)$, $f_k(0) > 0$. Moreover, since $h > 1$, by (13) we have $f_h(t_1) = 0$ and $f_h(t_h) = 1$. By Lemma 2.1, we conclude that f_h is not min-convex, a contradiction. Thus $t_1 = 0$. A similar argument shows $t_n = 1$. Summing up, there exist $0 = t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$ such that (13) holds. It immediately follows that *a*) holds, too. In order to prove *b*), *c*), and *d*) let us fix an interval $[t_i, t_i + 1]$, for $i = 1, \ldots, n-1$.

To prove *b*), suppose by way of contradiction that there exists $j \neq i, i+1$ such that $f_j(x) > 0$ for some $x \in [t_i, t_{i+1}]$. Say $j < i$. Since, by (13), $x \neq t_i, t_{i+1}$, we have that, on $t_j < t_i < x$, f_j takes values $f_j(t_j) = 1$, $f_j(t_i) = 0$, $f_j(x) > 0$. By Lemma 2.1, f_j is not min-convex, a contradiction. The argument for $j > i$ is analogous, and condition *b*) is proved.

From *b*) and the hypothesis that P is Ruspini, we immediately obtain *c*).

It remains to prove *d*). By (13), $f_i(t_i) = f_{i+1}(t_{i+1}) = 1$ and $f_i(t_{i+1}) = f_{i+1}(t_i) = 0$. Moreover, since f_i and f_{i+1} are strongly normal, and P is Ruspini, using *b*) we have

$$0 < f_i(x), f_{i+1}(x) < 1\,,\ \text{for all } x \in (t_i, t_{i+1})\,. \tag{14}$$

Since f_i, f_{i+1} are continuous, by the intermediate value theorem they are surjective when restricted to $[t_i, t_{i+1}]$. Suppose now that there exist $y < z \in (t_i, t_{i+1})$ such that $f_i(y) = f_i(z)$ (*absurdum hypothesis*). Observe that, by (14), $[y, z]$ is contained in the support of f_i and f_{i+1}. Pick $w \in (y, z)$. If $f_i(w) \leq f_i(y)$, then, by Lemma 2.1, f_i is not strictly min-convex on its support, a contradiction. If $f_i(w) > f_i(y)$, then, by *c*),

$$f_{i+1}(w) = 1 - f_i(w) < 1 - f_i(y) = f_{i+1}(y) = f_{i+1}(z)\,.$$

Thus f_{i+1} is not strictly min-convex on its support, a contradiction. Therefore, f_i and f_{i+1} are injective, and *d*) holds. □

4. *Intermezzo*: Łukasiewicz logic.

Łukasiewicz (*infinite-valued propositional*) *logic* is a non-classical many-valued system going back to the 1920's, cf. the early survey [14, §3], and its annotated English translation in [23, pp. 38–59]. The standard modern reference for Łukasiewicz logic is [5], while [19] deals with topics at the frontier of current research. Łukasiewicz logic can also be regarded as a member of a larger hierarchy of many-valued logics that was systematised by Petr Hájek in the late Nineties, cf. [13]. Let us recall some basic notions.

Let us fix once and for all the countably infinite set of propositional variables:

$$\textsc{Var} = \{X_1, X_2, \ldots, X_n, \ldots\}\,.$$

Let us write $\bot$ for the logical constant *falsum*, $\neg$ for the unary negation connective, and $\to$ for the binary implication connective. (Further derived connectives are introduced below.) The set $\textsc{Form}$ of (well-formed) formulæ[5] is defined exactly as in classical logic over the language $\{\bot, \neg, \to\}$.

The Łukasiewicz calculus is defined by the five[6] axiom schemata

(A0) $\bot \to \alpha$ (*Ex falso quodlibet.*)

(A1) $\alpha \to (\beta \to \alpha)$ (*A fortiori.*)

(A2) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ (Implication is transitive.)

(A3) $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$ (Disjunction is commutative.)

(A4) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$ (Contraposition.)

with *modus ponens* as the only deduction rule. Provability is defined exactly as in classical logic; $\vdash \alpha$ means that formula α is provable. We write $\mathscr{L}$ to denote Łukasiewicz logic.

The logical constant *verum* ($\top$), the conjunction ($\wedge$), the disjunction ($\vee$), and the biconditional ($\leftrightarrow$) are defined as in Table 1. From the definition of disjunction one sees that (A3) indeed asserts the commutativity of disjunction. Other common derived connectives are reported in the same table, with their definition. Some remarks are in order. Using the biconditional, one defines formulæ $\alpha, \beta \in \textsc{Form}$ to be *logically equivalent* just in case $\vdash \alpha \leftrightarrow \beta$ holds. The connectives $\odot$ and $\oplus$ are then De Morgan dual: $\alpha \oplus \beta$ is logically equivalent to $\neg(\neg\alpha \odot \neg\beta)$, and $\alpha \odot \beta$ is logically equivalent to $\neg(\neg\alpha \oplus \neg\beta)$. These connectives, known as the *strong disjunction* ($\oplus$) and *strong conjunction* ($\odot$) of $\mathscr{L}$, play a central rôle both in Hájek's treatment of many-valued logics [13], and in Chang's algebraisation of $\mathscr{L}$ via *MV-algebras* [5]. They are not idempotent, in the sense that $\alpha \oplus \alpha$ and α are not logically equivalent: only the implication $\alpha \to \alpha \oplus \alpha$ is provable; dual considerations apply to $\odot$. Conjunction ($\wedge$) and disjunction ($\vee$) also are De Morgan dual, but they are idempotent; in fact, they are sometimes called the *lattice connectives* because they induce the structure of a distributive

[5]A set of conventions for omitting parentheses in formulæ is usually adopted, and later extended to derived connectives. We do not spell the details here, as the conventions are analogous to the ones in classical logic, and are unlikely to cause confusion.

[6]In [5, Chapter 4] the language has no logical constants, and consequently (A0) does not appear as an axiom. We prefer to explicitly have $\bot$ in the language, and thus we add *Ex falso quodlibet* to the standard axiomatisation.

Notation	Definition	Name	Idempotent
$\bot$	–	*Falsum*	–
$\top$	$\neg\bot$	*Verum*	–
$\neg\alpha$	–	Negation	–
$\alpha \to \beta$	–	Implication	–
$\alpha \vee \beta$	$(\alpha \to \beta) \to \beta$	(Lattice) Disjunction	Yes
$\alpha \wedge \beta$	$\neg(\neg\alpha \vee \neg\beta)$	(Lattice) Conjunction	Yes
$\alpha \leftrightarrow \beta$	$(\alpha \to \beta) \wedge (\beta \to \alpha)$	Biconditional	–
$\alpha \oplus \beta$	$\neg\alpha \to \beta$	Strong disjunction	No
$\alpha \odot \beta$	$\neg(\alpha \to \neg\beta)$	Strong conjunction	No
$\alpha \ominus \beta$	$\neg(\alpha \to \beta)$	But not, or Difference	–

Table 1: Connectives in Łukasiewicz logic.

lattice in the algebraic semantics of $\mathscr{L}$. Finally, the connective $\ominus$ is the co-implication, *i.e.* the dual to $\to$.

If $S \subseteq \textsc{Form}$ is any set of formulæ, one writes $S \vdash \alpha$ to mean that α is provable in Łukasiewicz logic, under the additional set of assumptions S. When this is the case, one says that α is a *syntactic consequence* of S. Since each one of (A0–A4) is a principle of classical reasoning, and since *modus ponens* is a classically valid rule of inference, each formula provable in $\mathscr{L}$ is a theorem of classical propositional logic. The converse is not true: most notably, it is not hard to show that the *tertium non datur* law, $\alpha \vee \neg\alpha$, is not provable in Łukasiewicz logic. In fact, it can be shown that the addition of $\alpha \vee \neg\alpha$ as a sixth axiom schema to (A0–A4) yields classical logic.

By a *theory* in Łukasiewicz logic one means any set of formulæ that is closed under provability, *i.e.* is deductively closed. For any $S \subseteq \textsc{Form}$, the smallest theory that extends S exists: it is the *deductive closure* $S^{\vdash}$ of S, defined by $\alpha \in S^{\vdash}$ if, and only if, $S \vdash \alpha$. A theory Θ is *consistent* if $\Theta \neq \textsc{Form}$, and *inconsistent* otherwise. A theory Θ is *axiomatised by a set $S \subseteq \textsc{Form}$ of formulæ* if it so happens that $\Theta = S^{\vdash}$; and Θ is *finitely axiomatisable* if S can be chosen finite.

Let us now turn to the $[0,1]$-valued semantics. An *atomic assignment*, or *atomic evaluation*, is an arbitrary function $\overline{w}\colon \textsc{Var} \to [0,1]$. Such an atomic evaluation is uniquely extended to an *evaluation* of all formulæ, or *possible world*, *i.e.* to a function $w\colon \textsc{Form} \to [0,1]$, via the compositional rules:

$$\begin{aligned} w(\bot) &= 0\,, \\ w(\alpha \to \beta) &= \min\{1, 1-(w(\alpha)-w(\beta))\}\,, \\ w(\neg\alpha) &= 1 - w(\alpha)\,. \end{aligned}$$

It follows by trivial computations that the formal semantics of derived connectives is the one reported in Table 2. *Tautologies* are defined as those formulæ that evaluate to 1 under every evaluation. Let us write $\vDash \alpha$ to mean that the formula $\alpha \in \textsc{Form}$ is a tautology. The relativisation of this concept to theories

Notation	Formal semantics
$\bot$	$w(\bot) = 0$
$\top$	$w(\top) = 1$
$\neg\alpha$	$w(\neg\alpha) = 1 - w(\alpha)$
$\alpha \to \beta$	$w(\alpha \to \beta) = \min\{1, 1 - (w(\alpha) - w(\beta))\}$
$\alpha \vee \beta$	$w(\alpha \vee \beta) = \max\{w(\alpha), w(\beta)\}$
$\alpha \wedge \beta$	$w(\alpha \wedge \beta) = \min\{w(\alpha), w(\beta)\}$
$\alpha \leftrightarrow \beta$	$w(\alpha \leftrightarrow \beta) = 1 - \lvert w(\alpha) - w(\beta)\rvert$
$\alpha \oplus \beta$	$w(\alpha \oplus \beta) = \min\{1, w(\alpha) + w(\beta)\}$
$\alpha \odot \beta$	$w(\alpha \odot \beta) = \max\{0, w(\alpha) + w(\beta) - 1\}$
$\alpha \ominus \beta$	$w(\alpha \ominus \beta) = \max\{0, w(\alpha) - w(\beta)\}$

Table 2: Formal semantics of connectives in Łukasiewicz logic.

leads to the notion of semantic consequence. Let $S \subseteq \textsc{Form}$ be any subset, and let $\Theta = S^{\vdash}$ be its associated theory. Given $\alpha \in \textsc{Form}$, the assertion $S \vDash \alpha$ states that any evaluation $w\colon \textsc{Form} \to [0,1]$ that satisfies $w(S) = \{1\}$ — meaning that $w(\beta) = 1$ for each $\beta \in S$ — must also satisfy $w(\alpha) = 1$. When this is the case, we say that α is a *semantic consequence* of S. We write $S^{\vDash}$ for the set of semantic consequences of S.

It is an exercise to check that $\mathscr{L}$ enjoys the generalised validity theorem: for any $S \subseteq \textsc{Form}$ and any $\alpha \in \textsc{Form}$, if $S \vdash \alpha$ then $S \vDash \alpha$. (For a proof, see [5, 4.5.1].) On the other hand, it is a non-trivial theorem that $\mathscr{L}$ is complete[7] with respect to the many-valued semantics above: hence $\vdash \alpha$ if, and only if, $\vDash \alpha$, for any $\alpha \in \textsc{Form}$. The first proof of this appeared in [20]; see also [5, 4.5.1 & 4.5.2].

All of the above can be adapted in the obvious manner to the finite set $\textsc{Var}_n = \{X_1, \ldots, X_n\}$, in which case one speaks of Łukasiewicz logic *over n (propositional) variables*, denoted $\mathscr{L}_n$. The results in the sequel are formulated for $\mathscr{L}_n$, though they do admit extension to $\mathscr{L}$. Although, strictly speaking, one should introduce fresh consequence relation symbols $\vdash_n$ and $\vDash_n$ for $\mathscr{L}_n$, we will avoid this pedantry and use $\vdash$ and $\vDash$ instead. We will write $\textsc{Form}_n$ for the set of formulæ whose propositional variables are contained in $\textsc{Var}_n$.

5. From functions to logic: theories induced by fuzzy sets.

The following is a detailed definition of Θ_P as in (*).

Definition 5.1. (1) An assignment $\mu\colon \textsc{Form}_n \to [0,1]$ is *realised by P (at $x \in [0,1]$)* if $\mu(X_i) = f_i(x)$ for each $i = 1, \ldots, n$.

[7] However, $\mathscr{L}$ fails strong completeness (*i.e.* completeness for theories): there is a set $S \subseteq \textsc{Form}$ and a formula $\alpha \in \textsc{Form}$ such that $S \vDash \alpha$, but $S \nvdash \alpha$; see [5, 4.6].

(2) The *theory* $\Theta_P \subseteq \textsc{Form}_n$ *associated with* P is defined as the set of formulæ $\varphi \in \textsc{Form}_n$ such that $\mu(\varphi) = 1$ whenever the assignment $\mu\colon \textsc{Form}_n \to [0,1]$ is realised by P.

We record a simple fact for later use.

Lemma 5.1. *The set* $\Theta_P \subseteq \textsc{Form}_n$ *as in Definition 5.1 is indeed a theory, i.e. a deductively closed set of formulæ.*

Proof. For suppose $\varphi \in \textsc{Form}_n$ is such that $\Theta_P \vdash \varphi$. Since Łukasiewicz logic is sound with respect to $[0,1]$-valued assignments [5, 4.5.1], it follows that $\Theta_P \vDash \varphi$. If now the assignment μ is realised by P, by definition it evaluates to 1 each formula in Θ_P; from $\Theta_P \vDash \varphi$ we have $\mu(\varphi) = 1$, too, and therefore $\varphi \in \Theta_P$. Hence Θ_P is a theory. □

Remark 2. Observe that the theory Θ_P in Lemma 5.1 is always consistent: no assignment at all $\mu\colon \textsc{Form}_n \to [0,1]$ satisfies $\mu(\bot) = 1$, hence $\bot \notin \Theta_P$.

Remark 3. Notice that, as stated in the Prologue, Lemma 5.1 would hold (by the above proof) for any $[0,1]$-valued logic that satisfies the minimal requirement of soundness with respect to $[0,1]$-valued assignments. Also note that the finiteness of P plays no rôle in the proof. *In conclusion, any given collection of fuzzy sets gives rise to specific theory in any given* $[0,1]$*-valued logic.*

For an application of Remark 3 in the context of theories of vagueness, the interested reader may consult [16].

6. How to axiomatise a pseudo-triangular basis of fuzzy sets.

In this section we throughout work with Łukasiewicz logic over n propositional variables, $\mathscr{L}_n$. We prepare the following formulæ in $\textsc{Form}_n$.

$$\rho = X_1 \oplus X_2 \oplus \cdots \oplus X_n\,, \tag{15}$$

$$\alpha_{ij} = \neg(X_i \odot X_j)\,, \qquad \text{for } i,j = 1,\ldots,n, \text{ and } |i-j| = 1, \tag{16}$$

$$\beta_{ij} = \neg(X_i \wedge X_j)\,, \qquad \text{for } i,j = 1,\ldots,n, \text{ and } |i-j| > 1. \tag{17}$$

We further set

$$\mathbb{A} = \{\rho\} \cup \{\alpha_{ij} \mid i,j = 1,\ldots,n, \text{ and } |i-j| = 1\} \cup \\ \cup \{\beta_{ij} \mid i,j = 1,\ldots,n, \text{ and } |i-j| > 1\}\,.$$

By the 1-*set* of a formula $\varphi \in \textsc{Form}_n$ we mean the following subset of $[0,1]^n$:

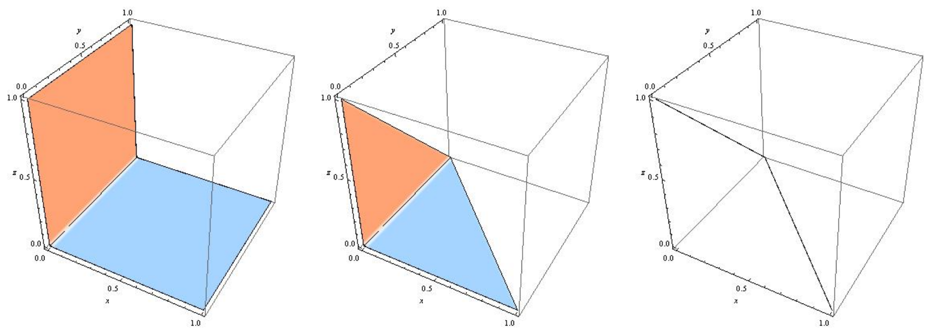
$$\{(x_1,\ldots,x_n) \in [0,1]^n \;\mid\; \mu_{\vec{x}}(\varphi) = 1\}\,,$$

where $\mu_{\vec{x}}\colon \textsc{Form}_n \to [0,1]$ is the unique evaluation extending the assignment $X_1 \mapsto x_1, \ldots, X_n \mapsto x_n$. The 1-set of a finite set of formulæ $\{\varphi_1,\ldots,\varphi_m\}$, moreover, is defined to be the 1-set of the formula $\varphi_1 \wedge \cdots \wedge \varphi_m$, or equivalently, the intersection of the 1-sets of φ_i, $i = 1,\ldots,m$.

To prove our Theorem II, the following lemma is needed.

Lemma 6.1. *The* 1*-set of the set of formulæ* $\mathbb{A}$ *is precisely the Hamiltonian path* $\bigcup_{i=1}^{n-1} \operatorname{Conv}\{e_i, e_{i+1}\}$.

Proof. For $n = 3$ the proof is provided by Figures 4(a), 4(b) and 4(c).



(a) 1-set of (17). (b) 1-set of (16–17). (c) 1-set of (15–17).

Figure 4: The case $n = 3$ of the proof of Lemma 6.1.

In general, let I_1 be the 1-set of (17). Then $(x_1, \dots, x_n) \in I_1$ if, and only if, for all $i, j \in \{1, \dots, n\}$ such that $|i - j| > 1$, $1 - \min\{x_i, x_j\} = 1$, that is, if, and only if, one between x_i and x_j equals 0. Thus, if F_i is the 2-dimensional face of $[0, 1]^n$ containing both e_i and e_{i+1}, we have $I_1 = \bigcup_{n-1}^{i=1} F_i$.

Let now I_2 be the 1-set of (16–17). Then, $(x_1, \dots, x_n) \in I_2$ if, and only if, $I_2 \subseteq I_1$, and, for all $i \in \{1, \dots, n-1\}$, $1 - \max\{0, x_i + x_{i+1} - 1\} = 1$, that is $x_i + x_{i+1} \leq 1$. Thus, $I_2 = \bigcup_{i=1}^{n-1} \operatorname{Conv}\{0, e_i, e_{i+1}\}$.

Finally, let I_3 be the 1-set of (15–17), *i.e.* of $\mathbb{A}$. Then, $(x_1, \dots, x_n) \in I_3$ if, and only if, $I_3 \subseteq I_2$, and $\min\{1, x_1 + \cdots + x_n\} = 1$. Thus, $I_3 = \bigcup_{i=1}^{n-1} \operatorname{Conv}\{e_i, e_{i+1}\}$. □

Theorem II. *The following are equivalent.*

i) *P is a pseudo-triangular basis of fuzzy sets.*

ii) *P is separating, and* $\Theta_P = \mathbb{A}^{\vdash}$.

Proof. *i)* ⇒ *ii)*. It is immediate to check that a pseudo-triangular basis of fuzzy sets is separating.

We next show that $\mathbb{A}^{\vdash} \subseteq \Theta_P$. To this aim, let $\mu\colon \mathrm{F{\small ORM}}_n \to [0, 1]$ be an assignment realised by P at x. By Definition 5.1:

$$\begin{aligned}
\mu(\rho) &= \min\{1, \mu(X_1) + \cdots + \mu(X_n)\} = \min\{1, f_1(x) + \cdots + f_n(x)\}\,; \\
\mu(\alpha_{ij}) &= 1 - \max\{0, \mu(X_i) + \mu(X_j) - 1\} = 1 - \max\{0, f_i(x) + f_j(x) - 1\}\,, \\
&\quad \text{for } i, j = 1, \dots, n, \text{ and } |i - j| = 1; \\
\mu(\beta_{ij}) &= 1 - \min\{\mu(X_i), \mu(X_j)\} = 1 - \min\{f_i(x), f_j(x)\}\,, \\
&\quad \text{for } i, j = 1, \dots, n, \text{ and } |i - j| > 1.
\end{aligned}$$

By c) in Definition 2.2, $\mu(\rho) = 1$, and $\mu(\alpha_{ij}) = 1$ for all $i, j = 1, \ldots, n$ such that $|i - j| = 1$. By b) in Definition 2.2, for $|i - j| > 1$, at least one between $f_i(x)$ and $f_j(x)$ equals 0. Thus, $\mu(\beta_{ij}) = 1$, for all $i, j = 1, \ldots, n$, and $|i - j| > 1$. Hence $\mathbb{A} \subseteq \Theta_P$, indeed. By Lemma 5.1 we therefore have $\mathbb{A}^{\vdash} \subseteq \Theta_P$.

It remains to prove that $\mathbb{A}^{\vdash} \supseteq \Theta_P$. Let $I_{\mathbb{A}}$ be the 1-set of $\mathbb{A}$. By Lemma 6.1 we have

$$I_{\mathbb{A}} = \bigcup_{i=1}^{n-1} \operatorname{Conv}\{e_i, e_{i+1}\}\,. \tag{18}$$

On the other hand, by Theorem 3 we have

$$\operatorname{ran} T_P = \bigcup_{i=1}^{n-1} \operatorname{Conv}\{e_i, e_{i+1}\}\,. \tag{19}$$

Hence $I_{\mathbb{A}} = \operatorname{ran} T_P$ by (18–19). If now $\varphi \in \Theta_P$, and I_φ is its 1-set, then each assignment realised by P at some point of $[0,1]$ satisfies φ by the definition of Θ_P, and therefore we have $I_\varphi \supseteq I_{\mathbb{A}}$. By the definition of semantic consequence we may rewrite the latter inclusion as $\varphi \in \mathbb{A}^{\vDash}$. Since $\mathbb{A}$ is a finite set, by the Hay-Wójcicki's Theorem [5, 4.6.7] we conclude $\mathbb{A}^{\vDash} = \mathbb{A}^{\vdash}$, as was to be shown.

ii) $\Rightarrow$ *i*) That P is separating is evidently equivalent to the fact that the map $T_P\colon [0,1] \to [0,1]^n$ is injective, so let us assume the latter for the rest of this proof. Writing again $I_{\mathbb{A}}$ for the 1-set of $\mathbb{A}$, by Lemma 6.1 we have (18). Hence it suffices to show

$$\operatorname{ran} T_P = I_{\mathbb{A}}\,, \tag{20}$$

for then Theorem I implies that P is a pseudo-triangular basis of fuzzy sets.

To prove the inclusion $\operatorname{ran} T_P \subseteq I_{\mathbb{A}}$, let $x = (x_1, \ldots, x_n) \in \operatorname{ran} T_P$. Then the assignment $\mu(X_i) = x_i$ is realised by P at x, and thus $\mu \vDash \Theta_P$ by the definition of Θ_P. Since $\Theta_P = \mathbb{A}^{\vdash}$ by assumption, and since $\mathbb{A}$ is finite, by the Hay-Wójcicki's Theorem [5, 4.6.7] we have $\Theta_P = \mathbb{A}^{\vDash}$, and therefore in particular $\mu \vDash \mathbb{A}$.

To prove the converse, let us set $R = \operatorname{ran} T_P \subseteq I_{\mathbb{A}}$. Assume by way of contradiction that $R \subset I_{\mathbb{A}}$, *i.e.* there is $x \in I_{\mathbb{A}}$ such that $x \notin R$. Therefore, if we set $D = I_{\mathbb{A}} \setminus \{x\}$, we have $R \subseteq D$. But since R is the continuous image of a connected set, namely $[0,1]$, it is itself connected, whereas by (18) we see that D has two connected components D_1 and D_n containing e_1 and e_n, respectively. Hence either $R \subseteq D_1$, or $R \subseteq D_n$. Say the former holds, without loss of generality, so that $e_n \notin R$. Next observe that R must be a closed set in the metric space $I_{\mathbb{A}}$, the latter endowed with the metric $d(\cdot,\cdot)$ induced by the Euclidean distance of $[0,1]^n$: indeed, this is a special case of the well-known closed map lemma, stating that a continuous map from a compact space to a Hausdorff (in particular, metric) space must be closed (=must send closed sets to closed sets). Hence e_n is an interior point of $I_{\mathbb{A}} \setminus R$, and thus there is an open set $U \equiv U(e_n, \epsilon) = \{x \in I_{\mathbb{A}} \mid d(e_n, x) < \epsilon\}$, for some real number $\epsilon > 0$, such that $U \cap R = \emptyset$. For an integer $k \geq 1$, let us consider the formula in $\textsc{Form}_n$

$$\varphi_k \;=\; \underbrace{\neg X_n \oplus \cdots \oplus \neg X_n}_{k \text{ times}}\,.$$

Further, let I_{φ_k} be the 1-set of φ_k. Direct inspection shows that $I_{\varphi_k} = \{(x_1, \ldots, x_n) \in [0,1]^n \mid x_n \leq \frac{k-1}{k}\}$. Let $k_0 \geq 1$ be the least integer that satisfies

$$k_0 \geq \frac{\sqrt{2}}{\epsilon}\,.$$

Then a simple computation shows

$$R \subseteq I_{\varphi_{k_0}}\,. \tag{21}$$

By (21) we infer at once

$$\varphi_{k_0} \in \Theta_P\,. \tag{22}$$

On the other hand, the assignment $\mu\colon \text{FORM}_n \to [0,1]$ such that $\mu(X_n) = 1$ and $\mu(X_i) = 0$, for $i = 1, \ldots, n-1$, satisfies $\mu(\varphi_{k_0}) = 0$ and evaluates each formula in $\mathbb{A}$ to 1, because $\{e_n\} \in I_{\mathbb{A}}$. Hence $\varphi_{k_0} \notin \mathbb{A}^{\vDash}$, and therefore

$$\varphi_{k_0} \notin \mathbb{A}^{\vdash} \tag{23}$$

by soundness [5, 4.5.1]. Now (22–23) yield the desired contradiction $\Theta_P \neq \mathbb{A}^{\vdash}$. □

7. Epilogue.

How can we generalise the results above to situations in which we are concerned with several physical observables? Here, we are to deal with fuzzy sets $f_i\colon [0,1]^m \to [0,1]$, $i = 1, \ldots, n$, the integer $m \geq 1$ being the number of observables. The generalisation of triangular bases to this setting requires elements of piecewise linear topology [21], which we assume in the following discussion. Consider a triangulation Σ of $[0,1]^m$, and let $v_1, \ldots, v_l$ be the (finite) list of vertices of Σ. For each v_i, let $h_i\colon [0,1]^m \to [0,1]$ be the function such that $h_i(v_i) = 1$, $h_i(v_j) = 0$ if $j \neq i$, and h_i agrees with an affine linear map $\mathbb{R}^m \to \mathbb{R}$ on each simplex of Σ. Then h_i is automatically continuous and piecewise-linear, and is called the *Schauder hat* of Σ at v_i. The collection $H_\Sigma = \{h_i \mid i = 1, \ldots, n\}$ is the *Schauder basis of* Σ. We then define the family P of fuzzy sets to be a *triangular basis* if is satisfies $P = H_\Sigma$ for some triangulation Σ of $[0,1]^n$. It is an exercise to check that this definition agrees with Definition 2.2 in case $n = 1$. It is also easy to see that Schauder bases are Ruspini partitions. Unfortunately, however, no elementary characterisation of Schauder bases analogous to our Theorem I is known. Nonetheless, abstract characterisations of Schauder bases have been obtained using homology and other mathematical tools [15]. This leads to the notion of *abstract Schauder bases*, the higher-dimensional analogue of pseudo-triangular bases of fuzzy sets, originally introduced in the last-named author's Ph.D. thesis. Remarkably, Łukasiewicz logic does express the notion of abstract Schauder basis, so that it is possible to formulate a higher-dimensional analogue of our Theorem II. For the algebraic treatment of abstract Schauder bases in the language of lattice-groups—structures closely related to MV-algebras, the algebraic semantics of Łukasiewicz logic—the interested reader is referred to [15, 18, 17], and to the references therein. For an account of bases in the context of MV-algebras and Łukasiewicz logic themselves, see [19].

Łukasiewicz and Gödel-Dummett logics are part of a hierarchy of systems based on *triangular norms*; see [13]. It has been argued that the hierarchy, together with its generalisations, provides a framework that makes precise the notion of *mathematical fuzzy logic* [6, 7]. The proof of Lemma 5.1 above, though easy, does say that the programme of axiomatising properties of fuzzy sets by means of a $[0, 1]$-valued logic makes sense at a very general level. It is important to stress that, to carry this programme out, one needs a reasonably complete set of analogues of standard notions in mathematical logic[8]—*e.g.* deductively closed theories and axiomatisations. Mathematical fuzzy logic, in the sense above, does provide such analogues. It is therefore possible, at least in principle, to develop this line of research extensively.[9] The benefits would surely be equally distributed between the theoretical and the application-oriented parties. One knows more about, say, Gödel-Dummett logic as a theoretical many-valued system, if one knows exactly to what extent the logic is capable of expressing the semantical notion of Ruspini partition. And one can make more conscious design choices in facing, say, the problem of developing a specific fuzzy-based control system, if one has that very same information about Gödel-Dummett logic available.

Acknowledgements. The present paper is a much-expanded follow up to the conference paper [10].

References

[1] S. Aguzzoli, M. Bianchi, V. Marra, A temporal semantics for basic logic, Studia Logica 92 (2009) 147–162.

[2] S. Aguzzoli, M. Busaniche, V. Marra, Spectral duality for finitely generated nilpotent minimum algebras, with applications, J. Logic Comput. 17 (2007) 749–765.

[3] S. Bova, P. Codara, D. Maccari, V. Marra, A logical analysis of Mamdani-type fuzzy inference, II: An experiment on the technical analysis of financial

[8] For the rôle that such notions may play even in developing a specific Mamdani-type fuzzy control system, see [4, 3].

[9] We already mentioned our previous contribution [9] in this direction. Here we add in passing that it would be important to further investigate systems that are not based on the three standard triangular norms (Łukasiewicz, Gödel, and Product). For an instance of how the lack of continuity in a triangular norm affects the formal semantics, see [2]. And for an example of how it may be more appropriate to use semantic not exclusively based on the notion of degree of truth, see [1].

markets, in: IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2010, pp. 1–8. 10.1109/FUZZY.2010.5584834.

[4] S. Bova, P. Codara, D. Maccari, V. Marra, A logical analysis of Mamdani-type fuzzy inference, I: Theoretical bases, in: IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2010, pp. 1–8. 10.1109/FUZZY.2010.5584830.

[5] R.L.O. Cignoli, I.M.L. D'Ottaviano, D. Mundici, Algebraic foundations of many-valued reasoning, volume 7 of *Trends in Logic—Studia Logica Library*, Kluwer Academic Publishers, Dordrecht, 2000.

[6] P. Cintula, P. Hájek, C. Noguera (Eds.), Handbook of Mathematical Fuzzy Logic, 1, volume 37 of *Studies in Logic – Mathematical Logic and Foundations*, College Publications, 2011.

[7] P. Cintula, P. Hájek, C. Noguera (Eds.), Handbook of Mathematical Fuzzy Logic, 2, volume 38 of *Studies in Logic – Mathematical Logic and Foundations*, College Publications, 2011.

[8] P. Codara, O.M. D'Antona, V. Marra, Best approximation of Ruspini partitions in Gödel logic, in: K. Mellouli (Ed.), Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU, 2007, volume 4724 of *Lecture Notes in Computer Science (LNAI)*, pp. 161–172.

[9] P. Codara, O.M. D'Antona, V. Marra, An analysis of Ruspini partitions in Gödel logic, Internat. J. Approx. Reason. 50 (2009) 825–836.

[10] P. Codara, O.M. D'Antona, V. Marra, A characterisation of bases of triangular fuzzy sets, in: IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2009, pp. 604–609.

[11] D. Dubois, H. Prade, Fuzzy sets and systems: theory and applications, volume 144 of *Mathematics in Science and Engineering*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.

[12] J. Dydak, Partitions of unity, in: Proceedings of the Spring Topology and Dynamical Systems Conference, 2003, volume 27 (1) of *Topology Proceedings*, pp. 125–171.

[13] P. Hájek, Metamathematics of fuzzy logic, volume 4 of *Trends in Logic—Studia Logica Library*, Kluwer Academic Publishers, Dordrecht, 1998.

[14] J. Łukasiewicz, A. Tarski, Untersuchngen über den Aussagenkalkül., C. R. Soc. Sc. Varsovie 23 (1930) 30–50.

[15] C. Manara, V. Marra, D. Mundici, Lattice-ordered abelian groups and Schauder bases of unimodular fans, Trans. Amer. Math. Soc. 359 (2007) 1593–1604 (electronic).

[16] V. Marra, The problem of artificial precision in theories of vagueness: a note on the rôle of maximal consistency, submitted to Erkenntnis (2011).

[17] V. Marra, Lattice-ordered abelian groups and Schauder bases of unimodular fans, II, Trans. Amer. Math. Soc (in press).

[18] V. Marra, D. Mundici, The Lebesgue state of a unital abelian lattice-ordered group, J. Group Theory 10 (2007) 655–684.

[19] D. Mundici, Advanced Łukasiewicz Calculus and MV-algebras, volume 35 of *Trends in Logic—Studia Logica Library*, Springer, New York, 2011.

[20] A. Rose, J.B. Rosser, Fragments of many-valued statement calculi, Trans. Amer. Math. Soc. 87 (1958) 1–53.

[21] C.P. Rourke, B.J. Sanderson, Introduction to piecewise-linear topology, Springer Study Edition, Springer-Verlag, Berlin, 1982. Reprint.

[22] E.H. Ruspini, A new approach to clustering, Information and Control 15 (1969) 22–32.

[23] A. Tarski, Logic, semantics, metamathematics. Papers from 1923 to 1938, Oxford at the Clarendon Press, 1956. Translated by J. H. Woodger.