Open Partitions and Probability Assignments in Gödel Logic

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Abstract. In the elementary case of finitely many events, we generalise to *Gödel* (*propositional infinite-valued*) *logic* — one of the fundamental fuzzy logics in the sense of Hájek — the classical correspondence between partitions, quotient measure spaces, and push-forward measures. To achieve this end, appropriate Gödelian analogues of the Boolean notions of probability assignment and partition are needed. Concerning the former, we use a notion of probability assignment introduced in the literature by the third-named author et al. Concerning the latter, we introduce and use *open partitions*, whose definition is justified by independent considerations on the relational semantics of Gödel logic (or, more generally, of the finite slice of intuitionistic logic). Our main result yields a construction of finite quotient measure spaces in the Gödelian setting that closely parallels its classical counterpart.

1 Introduction

We assume familiarity with Gödel (propositional infinite-valued) logic, one of the fundamental fuzzy logics in the sense of Hájek [4]; we recall definitions in Section 2.1 below. The problem of generalising elementary probability theory to such fuzzy logics has recently attracted considerable attention; let us mention e.g. [9] for Łukasiewicz logic, [8] for $[0, 1]$-valued logics with continuous connectives, and [2], [1] for Gödel logic. This paper falls into the same general research field.

Consider a finite set of (classical, yes/no) events $\mathscr{E}$, along with the finite Boolean algebras $B(\mathscr{E})$ that they generate. A *probability assignment* to $B(\mathscr{E})$ is a function $P\colon B(\mathscr{E}) \to [0, 1]$ satisfying *Kolmogorov's axioms*, namely,

(B1) $P(\top) = 1$, and
(B2) $P(X) + P(Y) = P(X \vee Y) + P(X \wedge Y)$ for all $X, Y \in B(\mathscr{E})$.

Here, $\top$ is the top element of $B(\mathscr{E})$, and $\vee$ and $\wedge$ denote the join and meet operation of $B(\mathscr{E})$, respectively. The assignment P is uniquely determined by its values on the atoms $A = \{a_1, \ldots, a_m\}$ of $B(\mathscr{E})$. In fact, there is a bijection between probability assignments to $B(\mathscr{E})$, and *probability distributions* on the set A, that is, functions $p\colon A \to [0, 1]$ such that

(BD) $\sum_{i=1}^{m} p(a_i) = 1$.

In one direction, one obtains such a p from a probability assignment P to $B(\mathscr{E})$ just as the restriction of P to A. Conversely, from a probability distribution p on A one obtains a probability assignment P to $B(\mathscr{E})$ by setting $P(X) = \sum_{a_i \leq X} p(a_i)$ for any event $X \in B(\mathscr{E})$. If one represents $B(\mathscr{E})$ as the Boolean algebra of subsets of A, this means that, for any $X \subseteq A$, one has $P(X) = \sum_{a_i \in X} p(a_i)$. In probabilistic parlance, one calls the set A a *sample space*, and its singleton subsets *elementary events*.

In several contexts related to probability theory, *partitioning* the sample space A is a process of fundamental importance. A *partition* of A is a collection of non-empty, pairwise disjoint subsets of A — often called *blocks* — whose union is A. A partition π of A can be regarded as a quotient object obtained from A. Indeed, there is a natural projection map $A \to \pi$ given by

$$a \in A \mapsto [a]_\pi \in \pi \ ,$$

where $[a]_\pi$ denotes the unique block of π that a belongs to. (Thus, $[a]_\pi$ is the equivalence class of a under the equivalence relation on A uniquely associated with π.) When, as is the present case, A carries a probability distribution, one would like each such quotient set π to inherit a unique probability distribution, too. This is indeed the case. Define a function $p_\pi \colon \pi \to [0,1]$ by

$$p_\pi([a]_\pi) = \sum_{a_i \in [a]_\pi} p(a_i) = P([a]_\pi) \ . \tag{1}$$

Then p_π is a probability distribution on the set π. To close a circle of ideas, let us return from the distribution p_π to a probability assignment to an appropriate algebra of events. For this, it suffices to observe that the partition $\pi = \{[a]_\pi \mid a \in A\}$ determines the unique subalgebra S_π of $B(\mathscr{E})$ whose atoms are given by $\bigvee [a]_\pi$, for $a \in A$. We can then define a probability assignment $P_{S_\pi} \colon S_\pi \to [0,1]$ starting from P via

$$P_{S_\pi}(X) = \sum_{[a]_\pi \leq X} P([a]_\pi) \ .$$

In other words, in the light of (1), P_{S_π} is the unique probability assignment to S_π that is associated with the probability distribution p_π on the atoms of S_π.

Although our current setting is restricted to finitely many events, and is thus elementary, generalisations of the standard ideas above play an important rôle in parts of measure theory. In particular, in certain contexts one constructs quotient measure spaces using as a key tool the push-forward measure along the natural projection map.[1] For our purposes, let $f \colon A \to B$ be a function between the finite sets A and B, and let $p \colon A \to [0,1]$ be a probability distribution on A. Define a map $p_f \colon B \to \mathbb{R}$ by setting

$$p_f(b) = \sum_{a \in f^{-1}(b)} p(a)$$

[1] For an influential account of these ideas, please see [10].

for any $b \in B$. We call p_f the *push-forward* of p along f. (Here, as usual, $p_f(b) = 0$ when the index set is empty.) One checks that p_f is again a probability distribution. If, moreover, f is a surjection, and thus induces a partition of A by taking fibres (=inverse images of elements in the codomain), then the following fact is easily verified.

Fact. *Let A be a finite set, π a partition of A, and $q\colon A \to \pi$ the natural projection map. Then, for every probability distribution $p\colon A \to [0,1]$ on A, the push-forward probability distribution p_q of p along q coincides with p_π in* (1).

Summing up, the fact above provides the desired construction of quotient measure spaces in the elementary case case of finitely many events. Our main result, Theorem 2 below, affords a generalisation of this construction to Gödel logic. To achieve this end, we need appropriate Gödelian analogues of the Boolean notions of probability assignment and partition. Concerning the former, we use a notion of probability assignment recently introduced in [2]; the needed background is in Subsection 2.3. Concerning the latter, in Section 3 we introduce *open partitions*, whose definition is justified by independent considerations on the relational semantics of Gödel logic. As a key tool for the proof of Theorem 2, we obtain in Theorem 1 a useful characterisation of open partitions.

2 Preliminary results, and background

2.1 Gödel logic

Equip the real unit interval $[0,1]$ with the operations $\wedge$, $\to$, and $\bot$ defined by

$$x \wedge y = \min(x,y)\,, \qquad x \to y = \begin{cases} 1 & \text{if } x \leq y\,, \\ y & \text{otherwise}\,, \end{cases} \qquad \bot = 0\ .$$

The tautologies of Gödel logics are exactly the formulas $\varphi(X_1,\ldots,X_n)$ built from connectives $\{\wedge,\to,\bot\}$ that evaluate constantly to 1 under any $[0,1]$-valued assignment to the propositional variables X_i, where each connective is interpreted as the operation denoted by the same symbol. As derived connectives, one has $\neg\varphi = \varphi \to \bot$, $\top = \neg\bot$, $\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. Thus, $\top$ is interpreted by 1, $\vee$ by maximum, and negation by

$$\neg x = \begin{cases} 1 & \text{if } x = 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Gödel logic can be axiomatised in the style of Hilbert with *modus ponens* as only deduction rule. In fact, completeness with respect to the many-valued semantics above can be shown to hold for arbitrary theories. For details, we refer to [4].

Gödel logic also coincides with the extension of the intuitionistic propositional calculus by the *prelinearity* axiom scheme $(\varphi \to \psi) \vee (\psi \to \varphi)$ (see again [4]). Thus, the algebraic semantics of Gödel logic is the well-known subvariety of

Heyting algebras[2] satisfying prelinearity, which we shall call *Gödel algebras*. By [6, Thm. 1], and in analogy with Boolean algebras, a finitely generated Gödel algebra is finite. Throughout, the operations of a Gödel algebra are always denoted by $\wedge$, $\vee$, $\to$, $\neg$, $\top$ (top element), and $\bot$ (bottom element).

2.2 Posets and open maps

For the rest of this paper, *poset* is short for partially ordered set, and all posets are assumed to be finite. If P is a poset (under the relation $\leq$) and $S \subseteq P$, the *lower set* generated by S is

$$\downarrow S = \{p \in P \mid p \leq s \text{ for some } s \in S\} \ .$$

(When S is a singleton $\{s\}$, we shall write $\downarrow s$ for $\downarrow \{s\}$.) A subposet $S \subseteq P$ is a *lower set* if $\downarrow S = S$. *Upper sets* and $\uparrow S$ are defined analogously. We write $\operatorname{Min} P$ and $\operatorname{Max} P$ for the set of minimal and maximal elements of P, respectively.

As we already mentioned, Gödel algebras are the same thing as Heyting algebras satisfying the prelinearity axiom. From any finite poset P one reconstructs a Heyting algebra, as follows. Let $\operatorname{Sub} P$ be the family of all lower sets of P. When partially ordered by inclusion, $\operatorname{Sub} P$ is a finite distributive lattice, and thus carries a unique Heyting implication adjoint to the lattice meet operation via *residuation*. Explicitly, if L is a finite distributive lattice, then its Heyting implication is given by

$$x \to y = \bigvee\{z \in L \mid z \wedge x \leq y\}$$

for all $x, y \in L$. Accordingly, we regard $\operatorname{Sub} P$ as a Heyting algebra.

Conversely, one can obtain a finite poset from any finite Heyting algebra H, by considering the poset $\operatorname{Spec} H$ of prime filters of H, ordered by reverse inclusion. Equivalently, one can think of $\operatorname{Spec} H$ as the poset of join-irreducible elements of H, with the order they inherit from H. (Let us recall that a *filter* of H is an upper set of H closed under meets; it is *prime* if it does not contain the bottom element of H, and contains either y or z whenever it contains $y \vee z$. We further recall that $x \in H$ is *join-irreducible* if it is not the bottom element of H, and whenever $x = y \vee z$ for $y, z \in H$, then either $x = y$ or $x = z$.)

The constructions of the two preceding paragraphs are inverse to each other, in the sense that for any finite Heyting algebra H one has an isomorphism of Heyting algebras

$$\operatorname{Sub} \operatorname{Spec} H \cong H \ . \tag{2}$$

In fact, the isomorphism (2) is natural. To explain this, let us recall that an order-preserving function $f : P \to Q$ between posets is called *open* if whenever $f(u) \geq v'$ for $u \in P$ and $v' \in Q$, there is $v \in P$ such that $u \geq v$ and $f(v) = v'$. From a logical point of view, if one regards P and Q as finite Kripke

[2] For background on Heyting algebras, see e.g. [7].

frames, then open maps are known as *p-morphisms*; cf. e.g. [3]. It is a folklore result that there is a categorical duality between finite Heyting algebras and their homomorphisms, and finite posets and open order-preserving maps between them. Given a homomorphism of finite Heyting algebras $h\colon A \to B$, the map $\mathrm{Spec}\, h\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ given by $\mathfrak{p} \mapsto h^{-1}(\mathfrak{p})$ (where $\mathfrak{p}$ is a prime filter of B) is open and order-preserving. Conversely, given an open order-preserving map $f\colon P \to Q$, the function $\mathrm{Sub}\, f\colon \mathrm{Sub}\, Q \to \mathrm{Sub}\, P$ given by $Q' \mapsto f^{-1}(Q')$ is a homomorphism of Heyting algebras. Specifically, the order-preserving property of f is equivalent to $\mathrm{Sub}\, f$ being a lattice homomorphism; and the additional assumption that f be open insures that $\mathrm{Sub}\, f$ preserves the Heyting implication, too. It can be checked that Spec and Sub (now regarded as functors) yield the aforementioned categorical duality.

Let us now restrict attention to finite *Gödel* algebras. A *forest* is a poset F such that $\downarrow x$ is totally ordered for any $x \in F$. In this case, it is customary to call $\mathrm{Min}\, F$ and $\mathrm{Max}\, F$ the sets of *roots* and *leaves* of F, respectively. Further, a lower set of the form $\downarrow x$, for $x \in F$, is called a *branch* of F. Note that any lower set of a forest F is itself a forest, and we shall call it a *subforest* of F. Horn proved [5, 2.4] that a Heyting algebra H is a Gödel algebra if and only if its prime filters are a forest under reverse inclusion, i.e. if $\mathrm{Spec}\, H$ is a forest. Using this fact, one sees that the categorical duality of the preceding paragraph restricts to a categorical duality between finite Gödel algebras with their homomorphisms, and forests with open maps between them. Since Gödel logic is a generalisation of classical logic, this duality has a Boolean counterpart as a special case. Namely, finite Boolean algebra and their homomorphisms are dually equivalent to finite sets and functions between them. To obtain this result starting from Gödel algebras, one just observes that a finite Gödel algebra G is Boolean if and only if $\mathrm{Spec}\, G$ is a forest consisting of roots only, that is, a finite set, and that an open map between such forests is just a set-theoretic function. Observe that the folklore duality between finite Boolean algebra and finite sets underlies the correspondence between probability assignments and distributions illustrated in the Introduction. Similarly, the folklore duality between forests and open maps will underlie the analogous correspondence for Gödel logic, which we will state in Proposition 1 below.

Example 1. If $G = \{\top, \bot\}$, then $\mathrm{Spec}\, G$ is a single point, the prime filter $\{\top\}$. If, even more trivially, G is the degenerate singleton algebra $G = \{\top = \bot\}$, then G has no prime filter at all, and thus $\mathrm{Spec}\, F$ is the empty forest. On the other hand, there is no such thing as a Gödel algebra with empty underlying set, because the signature contains the constant $\bot$. Next suppose G is the Gödel algebra whose Hasse diagram is depicted in Fig. 1. The join-irreducible elements of G are those labeled by X, $\neg X$, and $\neg\neg X$. Therefore, $\mathrm{Spec}\, G$ is the forest depicted in Fig. 2. To recover G from $\mathrm{Spec}\, F$, consider the collection $\mathrm{Sub}\,\mathrm{Spec}\, G$ of all lower sets of $\mathrm{Spec}\, G$ ordered by inclusion. This is depicted in Fig. 3. Ordering $\mathrm{Sub}\,\mathrm{Spec}\, G$ by inclusion, we get back (an algebra naturally isomorphic to) G. (Let us note that here, using algebraic terminology, G is the Gödel algebra freely generated by the generator X.)

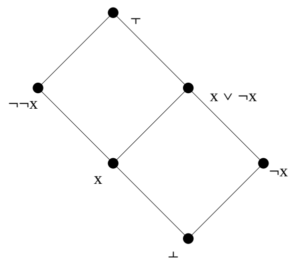

Fig. 1. A Gödel algebra G.

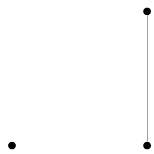

Fig. 2. The forest Spec G (cf. Fig. 1).

2.3 Probability assignments

Let G be a finite Gödel algebra. By a *probability assignment* to G we mean a function $P\colon G \to [0,1]$ such that, for any $X, Y, Z \in G$,

(G1) $P(\top) = 1$ and $P(\bot) = 0$,
(G2) $X \leq Y$ implies $P(X) \leq P(Y)$,
(G3) $P(X) + P(Y) = P(X \vee Y) + P(X \wedge Y)$ for all $X, Y \in G$, and
(G4) if X is covered[3] by Y that is covered by Z, and each X, Y, Z is either join-irreducible or coincides with $\bot$, then $P(X) = P(Y)$ implies $P(Y) = P(Z)$.

These axioms were first put forth in [2]. Clearly, (G1–3) are just standard properties of Boolean probability assignments: the first is normalisation; the second, monotonicity; the third, finite additivity. On the other hand, (G4) is characteristic of the Gödel case. If G happens to be a Boolean algebra, (G4) holds trivially, for in this case any two join-irreducible elements (=atoms) are incomparable. When G is not Boolean, then (G4) is an actual constraint on admissible distributions of values — a constraint arising from the nature of implication in Gödel logic. A further discussion of (G4) can be found in [2, Section III].

As in the Boolean case, there is a notion of probability distribution corresponding to (G1–G4). To define it, consider the forest $F = \operatorname{Spec} G$. A *probability distribution* on F is a function $p\colon F \to [0,1]$ such that

[3] This means that $X < Y$, and there is no element lying properly between X and Y.

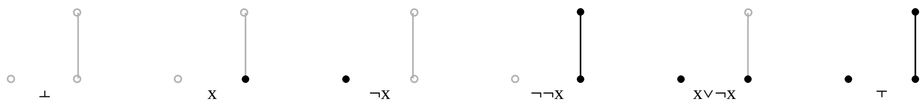

Fig. 3. The elements of Sub Spec G (cf. Fig. 1 and 2).

(GD1) $\sum_{x \in F} p(x) = 1$, and
(GD2) for all $x \leq y \in F$, $p(x) = 0$ implies $p(y) = 0$.

Axiom (GD2) is equivalent to the condition that $p^{-1}(0)$ be an upper set of F. Now, the following correspondence result holds.

Proposition 1. *Let G be a finite Gödel algebra. Without loss of generality, let us assume $G = \mathrm{Sub}\, F$ for a finite forest F. Let $\mathscr{P}$ be the family of all probability assignments to G, and $\mathscr{D}$ the family of all probability distributions on F. With each $P \in \mathscr{P}$, let us associate the map $p \colon F \to [0,1]$ such that, for each $x \in F$,*

$$p(x) = P(\downarrow x) - P(\downarrow x^{\lhd}) \ , \tag{3}$$

where $x^{\lhd}$ is the unique element of F that is covered by x, if $x \notin \mathrm{Min}\, F$, and $x^{\lhd} = \bot$, otherwise. Then the correspondence $p \mapsto P$ is a bijection between $\mathscr{P}$ and $\mathscr{D}$. Its inverse is given by

$$P(X) = \sum_{x \in X} p(x) \ , \tag{4}$$

for each $X \in G = \mathrm{Sub}\, F$.

For reasons of space, we shall omit the proof of Proposition 1. In the rest of the paper we shall find it expedient to work with distributions rather than assignments. Using Proposition 1 in a straightforward manner, it is possible to obtain a version of Theorem 2 below for probability assignments. Details are omitted, again due to space limitations.

3 Open partitions

We next introduce a key tool for the statement and proof of Theorem 2, namely, a notion of partition for forests.

Remark 1. It turns out that our results in this section apply equally well to (always finite) posets, with no additional complications. Therefore, here we shall work with posets and open maps between them. As mentioned in Subsection 2.2, in logical terms this amounts to working with finite Kripke frames and their p-morphisms, that is, with the relational semantics of the finite slice of intuitionistic logic. On the other hand, both in Proposition 1 and in Theorem 2 we restrict attention to forests only. Indeed, while the notion of open partition can be justified for the whole finite slice of intuitionistic logic, the notion of probability distribution given in (GD1–2) is intimately related to a complete $[0,1]$-valued semantics for the logic at hand — and, as is well known, no such complete semantics is available for full intuitionistic logic. We reserve a thorough discussion of these points for a future occasion.

In the Boolean case, a partition of a set A is the same thing as the collection of fibres of an appropriate surjection $f\colon A \to B$. Accordingly, we define as follows.

Definition 1. *An* open partition *of a poset P is a set-theoretic partition $\pi = \{B_1, \ldots, B_m\}$ of P that is induced by some surjective open map $f\colon P \to Q$ onto a poset Q. That is, for each $i = 1, \ldots, m$ there is $y \in Q$ such that*

$$B_i = f^{-1}(y) = \{x \in P \mid f(x) = y\} \ .$$

It follows that an open partition π of P carries an *underlying partial order*: namely, define

$$B_i \preceq B_j$$

if and only if

$$f(B_i) \leq f(B_j) \ \text{ in } \ Q \ .$$

It is easily verified that the order $\preceq$ does not depend on the choice of f and Q. Also note that π, regarded as a poset under $\preceq$, is order-isomorphic to (any choice of) Q.

Definition 1 has the advantage that it can be recast in quite general category-theoretic terms. However, it is also apparent that, in practice, it is quite inconvenient to work with — one needs to refer to f and Q, whereas in the Boolean case one has at hand the usual definition in terms of non-empty, pairwise disjoint subsets. This initial drawback is fully remedied by our main result on open partitions.

Theorem 1. *Let P be a poset, and let $\pi = \{B_1, \ldots, B_m\}$ be a set-theoretic partition of P. Then π is an open partition of P if and only if for each $B_i \in \pi$ there exist $i_1, i_2, \ldots, i_t \in \{1, \ldots, m\}$ such that*

$$\uparrow B_i = B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_t} \ . \tag{5}$$

In this case, the underlying order $\preceq$ of π is uniquely determined by

$$B_i \preceq B_j \ \textit{ iff } \ B_j \subseteq \uparrow B_i \ \textit{ iff there are } \ x \in B_i, y \in B_j \ \textit{with} \ x \leq y \,,$$

for each $B_i, B_j \in \pi$.

Proof. Suppose π is an open partition of P. By Definition 1 there exists a surjective open map f from P onto a poset Q whose set of fibres is π. Suppose, by way of contradiction, that (5) does not hold. Thus, there exist $p, q \in B_j$ such that $p \in \uparrow B_i$, but $q \notin \uparrow B_i$, for some $B_i, B_j \in \pi$. Let $f(B_i) = y$. Since f is order-preserving, $y \in \downarrow f(p)$. Since f is open, $y \notin \downarrow f(q)$, for else we would find $x \in B_i$ with $x \leq q$. But $f(q) = f(p)$ and we have a contradiction.

Suppose now that π satisfies (5). Endow π with the relation $\preceq$, defined as in Theorem 1. Observe that under the condition (5), for each $B_i, B_j \in \pi$, $B_j \subseteq \uparrow B_i$ if and only if there are $x \in B_i, y \in B_j$ with $x \leq y$. Indeed, whenever $x \leq y$, the block B_j intersects the upper set of the block B_i. By (5), B_j must be entirely contained in $\uparrow B_i$. The converse is trivial.

We show that $\preceq$ is a partial order on π. One can immediately check that $\preceq$ is reflexive and transitive. Let $B_i, B_j \in \pi$ be such that $B_i \preceq B_j$ and $B_j \preceq B_i$. Let $x \in B_i$. Since $B_j \preceq B_i$ there exists $y \in B_j$ such that $y \leq x$. Since $B_i \preceq B_j$ there exists $z \in B_i$ such that $z \leq y \leq x$. Iterating, since P is finite, we will find $p \in B_i$ and $q \in B_j$ satisfying $p \leq q \leq p$. Since π is a partition, we obtain $B_i = B_j$. Thus, the relation $\preceq$ is antisymmetric, and it is a partial order on π.

Let us consider now the projection map $f : P \to \pi$ which sends each element of P to its block. Let $x \in B_i$, $y \in B_j$, for $B_i, B_j \in \pi$. If $x \leq y$ then $f(x) = B_i \preceq f(y) = B_j$ and f is order-preserving. By construction, since π does not have empty blocks, f is surjective. To show f is open, we consider $u \in P$, $f(u) = B_t$, and $B_s \preceq B_t$, for some $B_s \in \pi$. Since $B_t \subseteq \uparrow B_s$, there exists $v \in B_s$ such that $v \leq u$. Since $f(v) = B_s$, f is open.

It remains to show that the last statement holds. Endow π with a partial order $\preceq'$ different from $\preceq$ and consider the map $f' : P \to \pi$ that sends each element of P to its own block. We consider two cases.

(Case 1). There exist $B_i, B_j \in \pi$ such that $B_i \preceq B_j$, but $B_i \npreceq' B_j$. Since there are $x \in B_i, y \in B_j$ with $x \leq y$, f' is not order-preserving.

(Case 2). There exist $B_i, B_j \in \pi$ such that $B_i \preceq' B_j$, but $B_i \npreceq B_j$. Let $y \in B_j$. By the definition of $\preceq$, for every $x \in B_i$, $x \nleq y$. Thus, f' is not an open map.

Thus, if one endows π with an order different from $\preceq$, one cannot find any surjective open map from P to π which induces the partition π. We therefore conclude that the order on π is uniquely determined, and the proof is complete. □

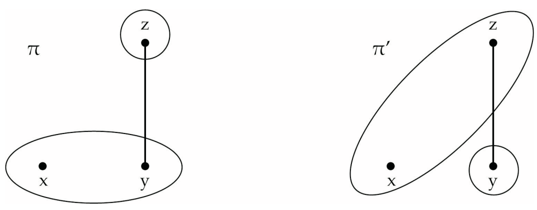

Fig. 4. Two set-theoretic partitions of a forest.

Example 2. We consider two different set-theoretic partitions $\pi = \{\{x, y\}, \{z\}\}$, and $\pi' = \{\{x, z\}, \{y\}\}$ of the same forest F. The partitions are depicted in Figure 4. It is immediate to check, using Condition (5) in Theorem 1, that π is an open partition of F, while π' is not.

4 Main Result

In Table 1, we summarise the correspondence between the fragments of the probability theory for Gödel logic sketched in the above, and the classical elementary theory. To state and prove our main result, we need one more definition to generalise the push-forward construction from finite sets to forests. Let $f: F_1 \to F_2$ be an open map between forests, and let $p: F_1 \to [0, 1]$ be a probability distribution. The *push-forward* of p along f is the function $p_f: F_2 \to \mathbb{R}$ defined by setting

Concept	**Boolean model**	**Gödelian model**
Sample space	Set	Forest
Event	Subset	Subforest
Elementary event	Singleton	Branch
Partition	Set-theoretic partition	Open partition
Structure of events	Boolean algebra of sets	Gödel algebra of forests
Probability assignment	Function satisfying (B1–2)	Function satisfying (G1–4)
Probability distribution	Function satisfying (BD2)	Function satisfying (GD1–2)

Table 1. Gödelian analogues of Boolean concepts.

$$p_f(y) = \sum_{x \in f^{-1}(y)} p(x)$$

for any $y \in F_2$.

Theorem 2. *Let F be a forest, π an open partition of F, and $q: F \to \pi$ the natural projection map, Then, for any probability distribution $p: F \to [0, 1]$, the push-forward p_q of p along q is again a probability distribution on π.*

Proof. It is clear that p_q takes values in the non-negative real numbers, because p does. Thus we need only prove that p_q satisfies (GD1–2).

We first prove that (GD1) holds. Let us display the given open partition as $\pi = \{B_1, \ldots, B_m\}$, and let us write $\preceq$ for its underlying order, and $\prec$ for the corresponding strict order. By definition, we have

$$p_q(B_i) = \sum_{x \in q^{-1}(B_i)} p(x) \tag{6}$$

for each $B_i \in \pi$. Since $q: F \to \pi$ is the natural projection map onto π, (6) can be rewritten as

$$p_q(B_i) = \sum_{x \in B_i} p(x) \tag{7}$$

Summing (7) over $i = 1, \ldots, m$, we obtain

$$\sum_{i=1}^{m} p_q(B_i) = \sum_{i=1}^{m} \sum_{x \in B_i} p(x) \ . \tag{8}$$

Since π is, in particular, a set-theoretic partition of F, from (8) we infer

$$\sum_{i=1}^{m} p_q(B_i) = \sum_{x \in F} p(x) = 1 \ , \tag{9}$$

with the latter equality following from the fact that p satisfies (GD1). This proves that p_q satisfies (GD1), too.

To prove (GD2), suppose, by way of contradiction, that $p_q^{-1}(0)$ is not an upper set — in particular, it is not empty. Then there exist $B_i \neq B_j \in \pi$ with

$$p_q(B_i) = 0 \ , \tag{10}$$

but

$$B_i \prec B_j \tag{11}$$

and

$$p_q(B_j) > 0 \ . \tag{12}$$

From (11), together with Theorem 1 and the fact that $B_i \cap B_j = \emptyset$, we know that there exist $x_i \in B_i$ and $x_j \in B_j$ such that

$$x_i < x_j \ .$$

From (10), along with (7) and the fact that p has non-negative range, we obtain

$$p_q(x) = 0 \text{ for all } x \in B_i \ . \tag{13}$$

By precisely the same token, from (12) we obtain that there exists an element $x'_j \in B_j$ such that

$$p_q(x'_j) > 0 \ . \tag{14}$$

In (14) we possibly have $x'_j \neq x_j$. However, we make the following

Claim. There exists $x'_i \in B_i$ with $x'_i < x'_j$.

Proof. By way of contradiction, suppose not. Then, writing $B_i = \{x_{i_1}, \ldots, x_{i_u}\}$, we have

$$x'_j \notin \ (\uparrow x_{i_1}) \cup \cdots \cup (\uparrow x_{i_u})$$

But, clearly,

$$(\uparrow x_{i_1}) \cup \cdots \cup (\uparrow x_{i_u}) = \uparrow B_i \ ,$$

so that

$$x'_j \notin \uparrow B_i \ .$$

Since, however, $x'_j \in B_j$, the latter statement immediately implies

$$B_j \nsubseteq \uparrow B_i \ .$$

Since, moreover, $B_i \prec B_j$ by (11), this contradicts Theorem 1. The Claim is settled. □

Now the Claim, together with (13–14), amounts to saying that $p^{-1}(0)$ is not an upper set, contradicting the assumption that p satisfies (GD2). Thus, p_q satisfies (GD2), too. This completes the proof. □

Example 3. We refer to the forest F and its open partition $\pi = \{\{x, y\}, \{z\}\}$ depicted in Figure 4. Consider the probability distribution $p : F \to [0, 1]$ such that $f(x) = 1$, and $f(y) = f(z) = 0$. Let $q : F \to \pi$ be the natural projection map.

The push-forward p_q of p along q is again a probability distribution on π. Indeed, $p_q(\{x, y\}) = 1$ and $p_q(\{z\}) = 0$, and thus p_q satisfies (GD1-2).

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