A SIMPLE COMBINATORIAL INTERPRETATION OF CERTAIN GENERALIZED BELL AND STIRLING NUMBERS

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Abstract. In a series of papers, P. Blasiak et al. developed a wide-ranging generalization of Bell numbers (and of Stirling numbers of the second kind) that is relevant to the so-called *Boson normal ordering problem*. They provided a recurrence and, more recently, also offered a (fairly complex) combinatorial interpretation of these numbers. We show that by restricting the numbers somewhat (but still widely generalizing Bell and Stirling numbers), one can supply a much more natural combinatorial interpretation. In fact, we offer two different such interpretations, one in terms of graph colourings and another one in terms of certain labelled Eulerian digraphs.

1. Introduction

In [BPS03a, BPS03b, MBP05, BHP$^+$07] P. Blasiak et al. introduced coefficients $B_{r,s}(n)$, and $S_{r,s}(n,k)$ that provide a wide-ranging generalization of Bell numbers, and of Stirling numbers of the second kind, respectively. In particular they defined the generalized Bell polynomial (see [BPS03a, Equations (1.5) and (2.1)])[1]

$$\begin{aligned} B_{r,s}(n,t) &= \sum_{k=s}^{ns} S_{r,s}(n,k)t^k \quad = \\ &= e^{-t}\sum_{k=0}^{\infty}\frac{1}{k!}\prod_{j=1}^{n}((k+(j-1)(r-s))_s t^k, \end{aligned} \tag{1}$$

where r, s, n, k are positive integers and $r \geq s$.

These coefficients generalize Bell numbers, and Stirling numbers of the second kind, usually denoted B_n, and $S(n,k)$, respectively, because by letting $r = s = t = 1$ in the above formula, one obtains the classical formula of Dobinski [Com74]

$$B_{1,1}(n) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}. \tag{2}$$

In fact $B_n = B_{1,1}(n)$, and $S(n,k) = S_{1,1}(n,k)$.

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[1] We denote by $(x)_n$ the falling factorial $x(x-1)\cdots(x-n+1)$. (Note that the authors of [BPS03a, BPS03b, MBP05, BHP$^+$07] use the symbol $x^{\underline{n}}$ instead.)

The work of P. Blasiak et al. was motivated by the fact that their coefficients are relevant for the so called *Boson normal ordering problem* [BPS03a, BPS03b, MBP05, BHP+07]; cf. also [BF11].

In [BPS03b] the authors asked for a combinatorial interpretation of these coefficients. Later on, in [MBP05], they provided one such interpretation, in terms of what they called *colonies of bugs*. We refer to [MBP05, Section III] for the exact definition, but we remark that a colony of bugs is a fairly complex object that corresponds to a labelled tree whose vertices include *labels* as well as *cells*. Each bug in a colony corresponds to a subtree, and has a *type* (r, s); it consists of a body of r cells, as well as of s *legs*, some of which can be *free* [MBP05, Section III]. It turns out ([MBP05, Theorem 3.1]) that $B_{r,s}(n)$ counts the number of colonies of n bugs each of type (r, s), and that $S_{r,s}(n, k)$ counts the number of such colonies having exactly k free legs.

In this note we suggest a simpler combinatorial interpretation of these coefficients, at least in some important cases. Our interpretations are stated in standard combinatorial terminology, in terms of colourings and labeled Eulerian digraphs.

Our focus is the case $r = s$. We supply two simple combinatorial interpretations of the coefficients $B_{m,m}(n)$ and $S_{m,m}(n, k)$, for all positive integers m, n, k. We note that these coefficients are still much more general than the Bell numbers $B_{1,1}(n)$ and the Stirling numbers of the second kind $S_{1,1}(n, k)$. Our first interpretation (Section 2) is in terms of colourings of a certain graph. In Sections 3 we supply another interpretation of the same numbers in terms of the number of certain labeled Eulerian digraphs. Finally, in Section 4 we remark that in the general case when r and s are different, there appear to be in certain cases well-known simple combinatorial interpretations as well; we discuss mostly the case $r = 2, s = 1$, but also remark on possible connections for certain values in the cases $r > 2$ and $s = 1$.

2. Colourings

A k-*colouring* of a graph G is a partition of the vertex set of G into k non-empty stable sets, *i.e.* sets not containing adjacent vertices. Each such stable set is called a *colour-class* of the partition.

Sometimes a k-colouring is defined as a mapping of vertices into a set of k colours, so that adjacent vertices obtain different colours. We note that for us the names of the colours do not play a role, *i.e.*, two mappings that yield the same partition are considered the same colouring. Moreover, we require that each colour-class is non-empty (which corresponds to the requirement that each colour is used).

We denote by K_m the complete graph on m vertices, and by nK_m the disjoint union of n copies of K_m.

For positive integers m, n, k, let $C_m(n, k)$ denote the number of k-colourings of nK_m. We first prove a recurrence for the numbers $C_m(n, k)$.

Proposition 2.1. *We have*

$$C_m(n,k) = \sum_{i=0}^{m} \binom{m}{i} (k-i)_{m-i} C_m(n-1, k-i) \,, \tag{3}$$

with initial conditions

$$C_m(n,k) = 0 \textit{ whenever } k < m\,, \quad \textit{and}$$

$$C_m(1,k) = \begin{cases} 1 & \textit{if } \ k = m\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

Proof. The case $k < m$ is trivial (with fewer then m colours we cannot colour K_m). It is also obvious that when $n = 1$ we have a unique k-colouring of K_m when $k = m$, and none when $k > m$.

To prove the recurrence, we describe how to obtain, in two steps, all k-colourings of nK_m, for $k \geq m$ and $n \neq 1$. Fix an arbitrary copy of K_m.

(1) Choose i vertices of the fixed K_m, each forming a singleton colour-class.

(2) Insert the remaining $m - i$ vertices of the fixed K_m in the colour-classes of all $(k - i)$-colourings of $(n - 1)K_m$.

Step (1) can be done in $\binom{m}{i}$ ways, and step (2) in $(k - i)_{m-i}C_m(n - 1, k - i)$ ways. Our claim is proved. □

Remark. Note that $C_m(n, nm) = 1$.

We now have the following result.

Proposition 2.2. *$S_{m,m}(n,k)$ counts the number of k-colourings of nK_m. In other words, $S_{m,m}(n,k) = C_m(n,k)$.*

Proof. A simple manipulation of the formulas shows that recurrence (3) coincides with the recurrence (21) in [BPS03b], namely:

$$S_{r,r}(n+1,k) = \sum_{p=0}^{r} \binom{k+p-r}{p} (r)_p S_{r,r}(n, k+p-r)\,.$$

Indeed, $(m)_i\binom{k+i-m}{i} = (k+i-m)_i\binom{m}{i}$. □

Needless to say, the recurrence (3) generalizes the classical recursion for the Stirling numbers of the second kind. Using (3), we can compute a few examples. In Table 1 we compute the number of k-colourings of nK_3.

$S_{3,3}(n,k)$	k=3	4	5	6	7	8	9	10
n=1	1							
2	6	18	9	1				
3	36	540	1242	882	243	27	1	
4	216	13608	94284	186876	149580	56808	11025	1107
5	1296	330480	6148872	28245672	49658508	41392620	18428400	4691412

TABLE 1. $S_{3,3}(n,k)$.

Table 1 also appears in [BPS03a, Table 1]. Denoting by $B_m(n)$ the number of *all* colourings of nK_m, we have (cf. [BPS03a, Equation (1.5)]):

$$B_m(n) = \sum_{k=m}^{nm} C_m(n,k) = \sum_{k=m}^{nm} S_{m,m}(n,k) = B_{m,m}(n)\,. \tag{4}$$

For instance, summing the rows of Table 1 we obtain $1, 34, 2971, 513559, \ldots$, that is the sequence $B_{3,3}(n)$ in [BPS03a], that is the sequence A069223 from [OEIS].

Example 1. Figure 1 shows the graph $2K_3$.

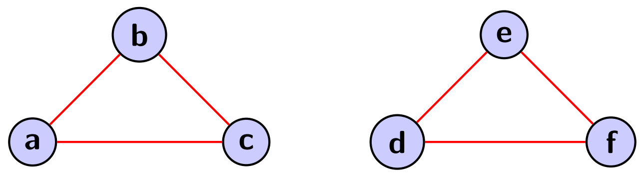

FIGURE 1. The graph $2K_3$.

The eighteen 4-colourings of $2K_3$ are

$$\begin{array}{cccccc} a|d|be|cf & a|d|bf|ce & a|e|bd|cf & a|e|bf|cd & a|f|bd|ce & a|f|be|cd \\ ad|b|e|cf & ad|b|f|ce & ae|b|d|cf & ae|b|f|cd & af|b|d|ce & af|b|e|cd \\ ad|be|c|f & ad|bf|c|e & ae|bd|c|f & ae|bf|c|d & af|bd|c|e & af|be|c|d\,. \end{array}$$

3. Labelled Eulerian Digraphs

We consider digraphs that allow loops and multiple edges in the same direction. A digraph G is *Eulerian* if at every vertex the in-degree equals the out-degree. (Note that we do not require G to be connected.) The edge set of an Eulerian digraph G can be partitioned into directed cycles. We call an Eulerian digraph (n, m)-*labelled* if its edge set is partitioned into n directed m-cycles, each with a distinguished first edge (and hence a unique second, third, etc., m-th edge). Figure 2 shows a $(2, 3)$-labelled Eulerian digraph, with its 2 directed 3-cycles; the j^{th} edge of the i^{th} cycle is labelled $e_{i,j}$.

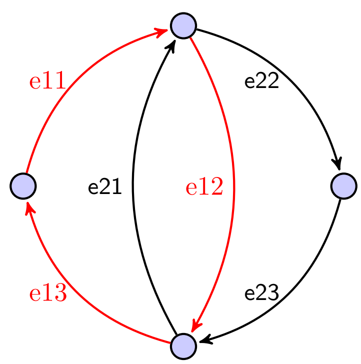

FIGURE 2. A $(2, 3)$-labelled Eulerian digraph.

Theorem 3.1. *The number of (n, m)-labelled Eulerian digraphs is equal to $B_{m,m}(n)$.*

Proof. We show a bijection between the set of (n, m)-labelled Eulerian digraphs and the number of colourings of nK_m. To this end we assign an arbitrary order to the n cliques of nK_m. Thus the vertices of nK_m will be called $v_{i,j}$ for $i = 1, 2, \ldots, n$, and $j = 1, 2, \cdots, m$. We define a bijective mapping ϕ associating $e_{i,j}$ with $v_{i,j}$. (Here $e_{i,j}$ is the i^{th} edge of the j^{th} cycle.)

• From graphs to colourings. Let $\mathcal{T}_m(n)$ be the set of (n, m)-labelled Eulerian digraphs. Here we establish a bijection between the k-colourings of nK_m and the elements of $\mathcal{T}_m(n)$ with k vertices. Let now τ be an element of $\mathcal{T}_m(n)$ with k

vertices. Let , for $t = 1, 2, \ldots, k$, B_t be the set of edges of τ that are incident in vertex t. It is obvious that

$$\{B_1, B_2, \ldots, B_k\}$$

is a partition of the set of edges of τ. Now, by construction, one sees that

$$\{\phi(B_1), \phi(B_2), \ldots, \phi(B_k)\}$$

is a k-colouring of nK_m.

For instance, the graph drawn in the picture 2 corresponds to the following colouring of $2K_3$

$$v_{1,3} \mid v_{1,1}v_{2,1} \mid v_{1,2}v_{2,3} \mid v_{2,2}$$

• From colourings to graphs. Let $\pi = \{B_1, B_2, ..., B_k\}$ be a colouring of nK_m. We describe the directed graph, τ, associated with π.

τ has k vertices, say $w_1, w_2, ..., w_k$. To define the edges of τ we assume first $m > 1$. Let, for $i = 1, 2, ..., n$, and $j = 1, 2, ..., m$, B_p be the block of π containing vertex $v_{i,j}$, and B_q be the block of π containing vertex $v_{i,j+1}$. Notice that the indices of the vertices of nK_m are considered in clockwise order: $v_{i,m+1} \equiv v_{i,1}$. Then edge $e_{i,j}$ starts at w_p, and ends at w_q.

If $m = 1$, the edges of τ are loops. Specifically $e_{i,1}$ starts and ends at w_t, where t is the index of the block of π containing vertex $v_{i,1}$. □

Thus we can say again that the number of (n, m)-labelled Eulerian digraphs with k vertices enjoy the same recurrence as $S_{m,m}(n, k)$. Therefore counting these graphs corresponds to another combinatorial interpretations of the coefficients of [BPS03a].

We close the Section with a remark. It is obvious that any k-colouring of a given set is fully described by any $k - 1$ of its colour-classes. Accordingly, one can give a slightly different interpretation of coefficients $S_{m,m}(n, k)$ by removing the last edge from each cycle, producing a partition into labeled directed paths instead of cycles. This model generalizes the concept of loopless, oriented multigraphs on n labeled arcs as in A020556 in [OEIS].

4. Conclusions

We hope that simpler combinatorial interpretations can be found for other generalized Bell numbers and Stirling numbers of the second kind. In particular, we note that our bijections (in Sections 2 and 3) exist for the disjoint union of cliques of different sizes. (Since the first version of this paper has appeared on arXiv, a combinatorial interpretation of $S_{r,s}(n, k)$ generalizing Proposition 2.2, by counting the number of k-colourings of more general graphs, has been provided by Engbers, Galvin, and Hilyard [EGH13]. In particular, it includes the case just mentioned of the disjoint union of cliques of different sizes.)

For the coefficients $S_{2,1}(n, k)$ we observe that Equation (15) of [BPS03b] implies that $S_{2,1}(n, k)$ is equal to the (positive) Lah number

$$L(n, k) = \frac{n!}{k!}\binom{n-1}{k-1}.$$

According to the classical interpretation of Lah numbers, this means that $S_{2,1}(n, k)$ counts the number of ordered placements of n balls into k boxes, and $B_{2,1}(n)$ counts the number of ordered placements of n balls into boxes [Com74].

Table 2 provides some values of $S_{2,1}(n,k)$. Those values also appear in [BPS03a, Table 1], and in sequences A105278 of [OEIS], where further combinatorial interpretations of such coefficients are proposed.

$S_{2,1}(n,k)$	k=1	2	3	4	5	6	7	8	9
n=1	1								
2	2	1							
3	6	6	1						
4	24	36	12	1					
5	120	240	120	20	1				
6	720	1800	1200	300	30	1			
7	5040	15120	12600	4200	630	42	1		
8	40320	141120	141120	58800	11760	1176	56	1	
9	362880	1451520	1693440	846720	211680	28224	2016	72	1

TABLE 2. $S_{2,1}(n,k)$.

Finally, we remark that the values of $S_{3,1}(n,1)$ in Table 1 in [BPS03a] appear to be identical to the sequence A001147 from [OEIS], which counts the number of increasing ordered rooted trees on $n+1$ vertices. (Here "increasing" means the vertices are labeled $0,1,2,...,n$ so that each path from the root has increasing labels.) Similarly, the values $S_{4,1}(n,1)$ appear to be identical to the sequence A007559 from [OEIS].

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