A SIMPLE COMBINATORIAL INTERPRETATION OF CERTAIN GENERALIZED BELL AND STIRLING NUMBERS

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Abstract. In a series of papers, P. Blasiak et al. developed a wide-ranging generalization of Bell numbers (and of Stirling numbers of the second kind) that is relevant to the so-called Boson normal ordering problem. They provided a recurrence and, more recently, also offered a (fairly complex) combinatorial interpretation of these numbers. We show that by restricting the numbers somewhat (but still widely generalizing Bell and Stirling numbers), one can supply a much more natural combinatorial interpretation. In fact, we offer two different such interpretations, one in terms of graph colourings and another one in terms of certain labelled Eulerian digraphs.

1. Introduction

In [BPS03a, BPS03b, MBP05, BHP+07] P. Blasiak et al. introduced coefficients $B_{r,s}(n)$, and $S_{r,s}(n,k)$ that provide a wide-ranging generalization of Bell numbers, and of Stirling numbers of the second kind, respectively. In particular they defined the generalized Bell polynomial (see [BPS03a, Equations (1.5) and (2.1)])

$$B_{r,s}(n,t) = \sum_{k=s}^{ns} S_{r,s}(n,k) t^k = e^{-t} \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{j=1}^{n} ((k + (j - 1)(r - s)) t^k,$$

where $r, s, n, k$ are positive integers and $r \geq s$.

These coefficients generalize Bell numbers, and Stirling numbers of the second kind, usually denoted $B_n$, and $S(n,k)$, respectively, because by letting $r = s = t = 1$ in the above formula, one obtains the classical formula of Dobinski [Com74]

$$B_{1,1}(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!},$$

In fact $B_n = B_{1,1}(n)$, and $S(n,k) = S_{1,1}(n,k)$.
The work of P. Blasiak et al. was motivated by the fact that their coefficients are relevant for the so called Boson normal ordering problem \([\text{BPS03a, BPS03b, MBP05, BHP+07; cf. also BF11.}]\).

In \([\text{BPS03b}]\) the authors asked for a combinatorial interpretation of these coefficients. Later on, in \([\text{MBP05}]\), they provided one such interpretation, in terms of what they called colonies of bugs. We refer to \([\text{MBP05, Section III}]\) for the exact definition, but we remark that a colony of bugs is a fairly complex object that corresponds to a labelled tree whose vertices include labels as well as cells. Each bug in a colony corresponds to a subtree, and has a type \((r,s)\); it consists of a body of \(r\) cells, as well as of \(s\) legs, some of which can be free \([\text{MBP05, Section III}]\). It turns out \(([\text{MBP05, Theorem 3.1}])\) that \(B_{r,s}(n)\) counts the number of colonies of \(n\) bugs each of type \((r,s)\), and that \(S_{r,s}(n,k)\) counts the number of such colonies having exactly \(k\) free legs.

In this note we suggest a simpler combinatorial interpretation of these coefficients, at least in some important cases. Our interpretations are stated in standard combinatorial terminology, in terms of colourings and labeled Eulerian digraphs.

Our focus is the case \(r = s\). We supply two simple combinatorial interpretations of the coefficients \(B_{m,m}(n)\) and \(S_{m,m}(n,k)\), for all positive integers \(m, n, k\). We note that these coefficients are still much more general than the Bell numbers \(B_{1,1}(n)\) and the Stirling numbers of the second kind \(S_{1,1}(n,k)\). Our first interpretation (Section 2) is in terms of colourings of a certain graph. In Sections 3 we supply another interpretation of the same numbers in terms of the number of certain labeled Eulerian digraphs. Finally, in Section 4 we remark that in the general case when \(r\) and \(s\) are different, there appear to be in certain cases well-known simple combinatorial interpretations as well; we discuss mostly the case \(r = 2, s = 1\), but also remark on possible connections for certain values in the cases \(r > 2\) and \(s = 1\).

### 2. Colourings

A \(k\)-colouring of a graph \(G\) is a partition of the vertex set of \(G\) into \(k\) non-empty stable sets, \(i.e.\) sets not containing adjacent vertices. Each such stable set is called a colour-class of the partition.

Sometimes a \(k\)-colouring is defined as a mapping of vertices into a set of \(k\) colours, so that adjacent vertices obtain different colours. We note that for us the names of the colours do not play a role, \(i.e.\), two mappings that yield the same partition are considered the same colouring. Moreover, we require that each colour-class is non-empty (which corresponds to the requirement that each colour is used).

We denote by \(K_m\) the complete graph on \(m\) vertices, and by \(nK_m\) the disjoint union of \(n\) copies of \(K_m\).

For positive integers \(m, n, k\), let \(C_m(n, k)\) denote the number of \(k\)-colourings of \(nK_m\). We first prove a recurrence for the numbers \(C_m(n, k)\).

**Proposition 2.1.** We have

\[
C_m(n, k) = \sum_{i=0}^{m} \binom{m}{i} (k-i)_{m-i} C_{m}(n-1, k-i),
\]
with initial conditions
\[ C_m(n,k) = 0 \text{ whenever } k < m, \quad \text{and} \]
\[ C_m(1,k) = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{otherwise}. \end{cases} \]

Proof. The case \( k < m \) is trivial (with fewer than \( m \) colours we cannot colour \( K_m \)).
It is also obvious that when \( n = 1 \) we have a unique \( k \)-colouring of \( K_m \) when \( k = m \),
and none when \( k > m \).

To prove the recurrence, we describe how to obtain, in two steps, all \( k \)-colourings
of \( nK_m \), for \( k \geq m \) and \( n \neq 1 \). Fix an arbitrary copy of \( K_m \).

1. Choose \( i \) vertices of the fixed \( K_m \), each forming a singleton colour-class.
2. Insert the remaining \( m - i \) vertices of the fixed \( K_m \) in the colour-classes of all \( (k - i) \)-colourings of \( (n - 1)K_m \).

Step (1) can be done in \( \binom{m}{i} \) ways, and step (2) in \( (k - i)^{m - i} C_m(n - 1, k - i) \) ways.
Our claim is proved. \( \Box \)

Remark. Note that \( C_m(n, nm) = 1 \).

We now have the following result.

**Proposition 2.2.** \( S_{m,m}(n,k) \) counts the number of \( k \)-colourings of \( nK_m \). In other
words, \( S_{m,m}(n,k) = C_m(n,k) \).

Proof. A simple manipulation of the formulas shows that recurrence (3) coincides
with the recurrence (21) in [BPS03b], namely:
\[ S_{r,r}(n + 1,k) = \sum_{p=0}^{r} \binom{k + p - r}{p} r_p S_{r,r}(n,k + p - r). \]
Indeed, \( \binom{m}{i} (k+i-m) \) \( \binom{m}{i} \).

Indeed, to say, the recurrence (3) generalizes the classical recursion for the
Stirling numbers of the second kind. Using (3), we can compute a few examples.
In Table 1 we compute the number of \( k \)-colourings of \( nK_3 \).

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Table 1. \( S_{3,3}(n,k) \).

Table 1 also appears in [BPS03a, Table 1]. Denoting by \( B_m(n) \) the number of
all colourings of \( nK_m \), we have (cf. [BPS03a, Equation (1.5)]):
\[ B_m(n) = \sum_{k=m}^{nm} C_m(n,k) = \sum_{k=m}^{nm} S_{m,m}(n,k) = B_{m,m}(n). \]

For instance, summing the rows of Table 1 we obtain 1, 34, 2971, 513559, ..., that
is the sequence \( B_{3,3}(n) \) in [BPS03a], that is the sequence A069223 from [OEIS].
Example 1. Figure 1 shows the graph $2K_3$.

![Figure 1. The graph $2K_3$.](image)

The eighteen 4-colourings of $2K_3$ are

- $a|d|be|cf$
- $a|d|bf|ce$
- $a|e|bd|cf$
- $a|e|bf|cd$
- $a|f|bd|ce$
- $a|f|be|cd$
- $ad|b|e|cf$
- $ad|b|f|ce$
- $ae|b|d|cf$
- $ae|b|f|cd$
- $af|b|d|ce$
- $af|b|e|cd$
- $ad|be|cf$
- $ad|bf|ce$
- $ae|bd|e|f$
- $ae|bf|cd$
- $af|bd|ce$
- $af|be|cd$

3. Labeled Eulerian Digraphs

We consider digraphs that allow loops and multiple edges in the same direction. A digraph $G$ is Eulerian if at every vertex the in-degree equals the out-degree. (Note that we do not require $G$ to be connected.) The edge set of an Eulerian digraph $G$ can be partitioned into directed cycles. We call an Eulerian digraph $(n,m)$-labelled if its edge set is partitioned into $n$ directed $m$-cycles, each with a distinguished first edge (and hence a unique second, third, etc., $m$-th edge). Figure 2 shows a $(2,3)$-labelled Eulerian digraph, with its 2 directed 3-cycles; the $j^{th}$ edge of the $i^{th}$ cycle is labelled $e_{i,j}$.

![Figure 2. A $(2,3)$-labelled Eulerian digraph.](image)

Theorem 3.1. The number of $(n,m)$-labelled Eulerian digraphs is equal to $B_{m,m}(n)$.

Proof. We show a bijection between the set of $(n,m)$-labelled Eulerian digraphs and the number of colourings of $nK_m$. To this end we assign an arbitrary order to the $n$ cliques of $nK_m$. Thus the vertices of $nK_m$ will be called $v_{i,j}$ for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$. We define a bijective mapping $\phi$ associating $e_{i,j}$ with $v_{i,j}$. (Here $e_{i,j}$ is the $i^{th}$ edge of the $j^{th}$ cycle.)

- From graphs to colourings. Let $\mathcal{T}_m(n)$ be the set of $(n,m)$-labelled Eulerian digraphs. Here we establish a bijection between the $k$-colourings of $nK_m$ and the elements of $\mathcal{T}_m(n)$ with $k$ vertices. Let now $\tau$ be an element of $\mathcal{T}_m(n)$ with $k$
vertices. Let, for \( t = 1, 2, \ldots, k \), \( B_t \) be the set of edges of \( \tau \) that are incident in vertex \( t \). It is obvious that
\[
\{B_1, B_2, \ldots, B_k\}
\]
is a partition of the set of edges of \( \tau \). Now, by construction, one sees that
\[
\{\phi(B_1), \phi(B_2), \ldots, \phi(B_k)\}
\]
is a \( k \)-colouring of \( nK_m \).

For instance, the graph drawn in the picture 2 corresponds to the following colouring of \( 2K_3 \)
\[
v_{1,3} \mid v_{1,1}v_{2,1} \mid v_{1,2}v_{2,3} \mid v_{2,2}
\]

- From colourings to graphs. Let \( \pi = \{B_1, B_2, \ldots, B_k\} \) be a colouring of \( nK_m \). We describe the directed graph, \( \tau \), associated with \( \pi \).

\( \tau \) has \( k \) vertices, say \( w_1, w_2, \ldots, w_k \). To define the edges of \( \tau \) we assume first \( m > 1 \). Let, for \( i = 1, 2, \ldots, n \), and \( j = 1, 2, \ldots, m \), \( B_p \) be the block of \( \pi \) containing vertex \( v_{i,j} \), and \( B_q \) be the block of \( \pi \) containing vertex \( v_{i,j+1} \). Notice that the indices of the vertices of \( nK_m \) are considered in clockwise order: \( v_{i,m+1} \equiv v_{i,1} \). Then edge \( e_{i,j} \) starts at \( w_p \), and ends at \( w_q \).

If \( m = 1 \), the edges of \( \tau \) are loops. Specifically \( e_{i,1} \) starts and ends at \( w_t \), where \( t \) is the index of the block of \( \pi \) containing vertex \( v_{i,1} \).

Thus we can say again that the number of \((n,m)\)-labelled Eulerian digraphs with \( k \) vertices enjoy the same recurrence as \( S_{m,m}(n,k) \). Therefore counting these graphs corresponds to another combinatorial interpretations of the coefficients of \[BPS03a\].

We close the Section with a remark. It is obvious that any \( k \)-colouring of a given set is fully described by any \( k-1 \) of its colour-classes. Accordingly, one can give a slightly different interpretation of coefficients \( S_{m,m}(n,k) \) by removing the last edge from each cycle, producing a partition into labeled directed paths instead of cycles. This model generalizes the concept of loopless, oriented multigraphs on \( n \) labeled arcs as in A020556 in \[OEIS\].

4. Conclusions

We hope that simpler combinatorial interpretations can be found for other generalized Bell numbers and Stirling numbers of the second kind. In particular, we note that our bijections (in Sections 2 and 3) exist for the disjoint union of cliques of different sizes. (Since the first version of this paper has appeared on arXiv, a combinatorial interpretation of \( S_{r,s}(n,k) \) generalizing Proposition 2.2, by counting the number of \( k \)-colourings of more general graphs, has been provided by Engbers, Galvin, and Hilyard \[EGH13\]. In particular, it includes the case just mentioned of the disjoint union of cliques of different sizes.)

For the coefficients \( S_{2,1}(n,k) \) we observe that Equation (15) of \[BPS03b\] implies that \( S_{2,1}(n,k) \) is equal to the (positive) Lah number
\[
L(n,k) = \frac{n!}{k! \binom{n-1}{k-1}}.
\]
According to the classical interpretation of Lah numbers, this means that \( S_{2,1}(n,k) \) counts the number of ordered placements of \( n \) balls into \( k \) boxes, and \( B_{2,1}(n) \) counts the number of ordered placements of \( n \) balls into boxes \[Com74\].
Table 2 provides some values of $S_{2,1}(n, k)$. Those values also appear in [BPS03a, Table 1], and in sequences A105278 of [OEIS], where further combinatorial interpretations of such coefficients are proposed.

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Table 2. $S_{2,1}(n, k)$.

Finally, we remark that the values of $S_{3,1}(n, 1)$ in Table 1 in [BPS03a] appear to be identical to the sequence A001147 from [OEIS], which counts the number of increasing ordered rooted trees on $n+1$ vertices. (Here "increasing" means the vertices are labeled 0, 1, 2, ..., $n$ so that each path from the root has increasing labels.) Similarly, the values $S_{4,1}(n, 1)$ appear to be identical to the sequence A007559 from [OEIS].

References


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