Independent subsets of powers of paths, and Fibonacci cubes

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Abstract

We provide a formula for the number of edges of the Hasse diagram of the independent subsets of the h^{th} power of a path ordered by inclusion. For $h = 1$ such a value is the number of edges of a Fibonacci cube. We show that, in general, the number of edges of the diagram is obtained by convolution of a Fibonacci-like sequence with itself.

Keywords: Independent subset, path, power of graph, Fibonacci cube.

1 Introduction

For a graph $\mathbf{G}$ we denote by $V(\mathbf{G})$ the set of its vertices, and by $E(\mathbf{G})$ the set of its edges.

Definition 1.1 *For $n, h \geq 0$, the* h-power of a path*, denoted by $\mathbf{P}_n^{(h)}$, is a graph with n vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{P}_n^{(h)})$ if and only if $|j - i| \leq h$.*

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Thus, for instance, $\mathbf{P}_n^{(0)}$ is the graph made of n isolated nodes, and $\mathbf{P}_n^{(1)}$ is the path with n vertices.

Definition 1.2 *An* independent subset of a graph $\mathbf{G}$ *is a subset of* $V(\mathbf{G})$ *not containing adjacent vertices.*

Notation. (i) We denote by $p_n^{(h)}$ the number of independent subsets of $\mathbf{P}_n^{(h)}$. (ii) We denote by $\mathbf{H}_n^{(h)}$ the Hasse diagram of the poset of independent subsets of $\mathbf{P}_n^{(h)}$ ordered by inclusion, and by $H_n^{(h)}$ the number of edges of $\mathbf{H}_n^{(h)}$.

In this work we evaluate $p_n^{(h)}$, and $H_n^{(h)}$. Our main result (Theorem 3.4) is that, for $n, h \geq 0$, the sequence $H_n^{(h)}$ is obtained by convolving the sequence $\underbrace{1, \ldots, 1}_{h}, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots$ with itself.

Clearly, $\mathbf{H}_n^{(0)}$ is the n-dimensional cube. Thus, on one hand, our work generalizes the known formula $n2^{n-1}$ for the number of edges of the Boolean lattice with n atoms, obtained by the convolution of the sequence $\{2^n\}_{n\geq 0}$ with itself. From a different perspective, this work could be seen as yet another generalization of the notion of Fibonacci cube. Indeed, observe that every independent subset S of $\mathbf{P}_n^{(h)}$ can be represented by a binary string $b_1 b_2 \cdots b_n$, where, for $i = 1, \ldots, n$, $b_i = 1$ if and only if $v_i \in S$. More specifically, each independent subset of $\mathbf{P}_n^{(h)}$ is associated with a binary string of length n such that the distance between any two 1's of the string is greater than h. For $h = 1$ the binary strings associated with independent subsets of $\mathbf{P}_n^{(h)}$ are *Fibonacci strings of order* n, and the Hasse diagram of the set of all such strings ordered bitwise is a *Fibonacci cube of order* n (see [5,7]). Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [3], and their combinatorial structure has been further investigated, *e.g.* in [6,7]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, *e.g.*, [4,5]). As far as we now, our generalization, described in terms of independent subsets of powers of paths ordered by inclusion, is a new one.

2 The independent subsets of powers of paths

We denote by $p_{n,k}^{(h)}$ the number of independent k-subsets of $\mathbf{P}_n^{(h)}$.

Lemma 2.1 *For* $n, h, k \geq 0$, $p_{n,k}^{(h)} = \binom{n-hk+h}{k}$.

Proof. See [2, Theorem 1], and [1], where we establish a bijection between independent k-subset of $\mathbf{P}_n^{(h)}$ and k-subsets of a set with $(n-hk+h)$ elements.$\square$

For $n, h \geq 0$, the number of all independent subsets of $\mathbf{P}_n^{(h)}$ is

$$p_n^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n-hk+h}{k}.$$

Remark 2.2 *Denote by F_n the n^{th} element of the Fibonacci sequence $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$.*

Lemma 2.3 *For $n, h \geq 0$,* $p_n^{(h)} = \begin{cases} n+1 & \text{if } n \leq h+1\,, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h+1\,. \end{cases}$

Proof. See the first part of [2, Proof of Theorem 1], or [1]. □

3 The poset of independent subsets of powers of paths

Figure 1 shows a few Hasse diagrams $\mathbf{H}_n^{(h)}$. Notice that, as mentioned in the introduction, for each n, $\mathbf{H}_n^{(1)}$ is a Fibonacci cube.

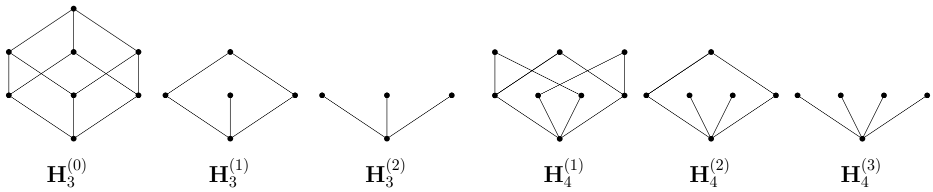

Fig. 1. Some $\mathbf{H}_n^{(h)}$.

Since in $\mathbf{H}_n^{(h)}$ each non-empty independent k-subset covers exactly k independent $(k-1)$-subsets, we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k p_{n,k}^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \binom{n-hk+h}{k}. \tag{1}$$

Let now $T_{k,i}^{(n,h)}$ be the number of independent k-subsets of $\mathbf{P}_n^{(h)}$ containing the vertex v_i, and let, for $h, k \geq 0$, $n \in \mathbb{Z}$, $\bar{p}_{n,k}^{(h)} = \begin{cases} p_{0,k}^{(h)} & \text{if } n < 0\,, \\ p_{n,k}^{(h)} & \text{if } n \geq 0\,. \end{cases}$

Lemma 3.1 *For $n, h, k \geq 0$, and $1 \leq i \leq n$,*

$$T_{k,i}^{(n,h)} = \sum_{r=0}^{k-1} \bar{p}_{i-h-1,r}^{(h)} \; \bar{p}_{n-i-h,k-1-r}^{(h)}.$$

Proof. No independent subset of $\mathbf{P}_n^{(h)}$ containing v_i contains any of the elements $v_{i-h}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+h}$. Let r and s be non-negative integers whose sum is $k-1$. Each independent k-subset of $\mathbf{P}_n^{(h)}$ containing v_i can be obtained by adding v_i to a $(k-1)$-subset $R \cup S$ such that

(a) $R \subseteq \{v_1, \ldots, v_{i-h-1}\}$ is an independent r-subset of $\mathbf{P}_n^{(h)}$;

(b) $S \subseteq \{v_{i+h+1}, \ldots, v_n\}$ is an independent s-subset of $\mathbf{P}_n^{(h)}$.

Viceversa, one can obtain each of this pairs of subsets by removing v_i from an independent k-subset of $\mathbf{P}_n^{(h)}$ containing v_i. Thus, $T_{k,i}^{(n,h)}$ is obtained by counting independently the subsets of type (a) and (b). Noting that the subsets of type (b) are in bijection with the independent s-subsets of $\mathbf{P}_{n-i-h}^{(h)}$, the lemma is proved. □

In order to obtain our main result, we prepare a lemma.

Lemma 3.2 *For positive n,*

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} .$$

Proof. The inner sum counts the number of k-subsets exactly k times, one for each element of the subset. That is, $\sum_{i=1}^{n} T_{k,i}^{(n,h)} = k p_{n,k}^{(h)}$. The lemma follows directly from Equation (1). □

Next we introduce a family of Fibonacci-like sequences.

Definition 3.3 *For $h \geq 0$, and $n \geq 1$, the h-*Fibonacci sequence *$\mathcal{F}^{(h)} = \{F_n^{(h)}\}_{n \geq 1}$ is the sequence whose elements are*

$$F_n^{(h)} = \begin{cases} 1 & \text{if } \ n \leq h+1 \,, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } \ n > h+1 . \end{cases}$$

From Lemma 2.3, and setting for $h \geq 0$, and $n \in \mathbb{Z}$, $\bar{p}_n^{(h)} = \begin{cases} p_0^{(h)} & \text{if } \ n < 0 \,, \\ p_n^{(h)} & \text{if } \ n \geq 0 \,, \end{cases}$
we have that,

$$F_i^{(h)} = \bar{p}_{i-h-1}^{(h)} \,, \quad \text{for each } \ i \geq 1 \,. \tag{2}$$

Thus, we can write $\mathcal{F}^{(h)} = \underbrace{1, \ldots, 1}_{h}, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots$.

In the following, we use the discrete convolution operation $*$, as follows.

$$\left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n) \doteq \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)} \,. \tag{3}$$

Theorem 3.4 *For $n, h \geq 0$, the following holds.*

$$H_n^{(h)} = \left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n).$$

Proof. The sum $\sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)}$ counts the number of independent subsets of $\mathbf{P}_n^{(k)}$ containing v_i. We can also obtain such a value by counting the independent subsets of both $\{v_1, \ldots, v_{i-h-1}\}$, and $\{v_{i+h+1}, \ldots, v_n\}$. Thus, we have:

$$\sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \bar{p}_{i-h-1}^{(h)} \, \bar{p}_{n-h-i}^{(h)}.$$

Using Lemma 3.2 we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \, \bar{p}_{n-h-i}^{(h)}.$$

By Equation (2) we have $\sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \, \bar{p}_{n-h-i}^{(h)} = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)}$. By (3), the theorem is proved. □

Further properties of coefficients $H_n^{(h)}$, and $p_n^{(h)}$ are discussed in [1]. Moreover, in [1] we investigate the case of powers of cycles, and its connection with Lucas cubes.

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