ON THE INDEPENDENT SUBSETS OF POWERS OF PATHS AND CYCLES

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Abstract. In the first part of this work we provide a formula for the number of edges of the Hasse diagram of the independent subsets of the h^{th} power of a path ordered by inclusion. For $h = 1$ such a value is the number of edges of a Fibonacci cube. We show that, in general, the number of edges of the diagram is obtained by convolution of a Fibonacci-like sequence with itself.

In the second part we consider the case of cycles. We evaluate the number of edges of the Hasse diagram of the independent subsets of the h^{th} power of a cycle ordered by inclusion. For $h = 1$, and $n > 1$, such a value is the number of edges of a Lucas cube.

1. Introduction

For a graph $\mathbf{G}$ we denote by $V(\mathbf{G})$ the set of its vertices, and by $E(\mathbf{G})$ the set of its edges.

Definition 1.1. For $n, h \geq 0$,

(i) the *h-power of a path*, denoted by $\mathbf{P}_n^{(h)}$, is a graph with n vertices v_1, v_2, $\ldots$, v_n such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{P}_n^{(h)})$ if and only if $|j - i| \leq h$;

(ii) the *h-power of a cycle*, denoted by $\mathbf{Q}_n^{(h)}$, is a graph with n vertices v_1, v_2, $\ldots$, v_n such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{Q}_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$.

Thus, for instance, $\mathbf{P}_n^{(0)}$ and $\mathbf{Q}_n^{(0)}$ are the graphs made of n isolated nodes, $\mathbf{P}_n^{(1)}$ is the path with n vertices, and $\mathbf{Q}_n^{(1)}$ is the cycle with n vertices. Figures 1(a), and 1(b) show some powers of paths and cycles, respectively.

Definition 1.2. An *independent subset of a graph* $\mathbf{G}$ is a subset of $V(\mathbf{G})$ not containing adjacent vertices.

Let $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ be the Hasse diagrams of the posets of independent subsets of $\mathbf{P}_n^{(h)}$, and $\mathbf{Q}_n^{(h)}$, respectively, ordered by inclusion. Clearly, $\mathbf{H}_n^{(0)} \cong \mathbf{M}_n^{(0)}$ is a Boolean lattice with n atoms (n-cube, for short).

Every independent subset S of $\mathbf{P}_n^{(h)}$ can be represented by a binary string $b_1 b_2 \cdots b_n$, where, for $i = 1, \ldots, n$, $b_i = 1$ if and only if $v_i \in S$. Specifically, each independent subset of $\mathbf{P}_n^{(h)}$ is associated with a binary string of length n such

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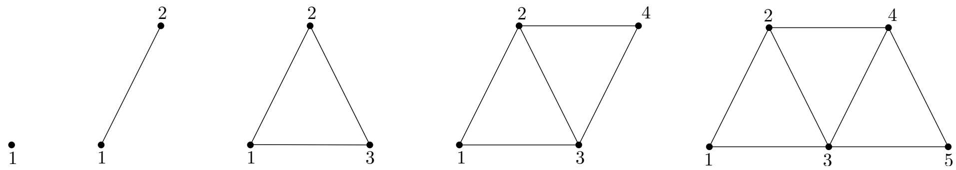
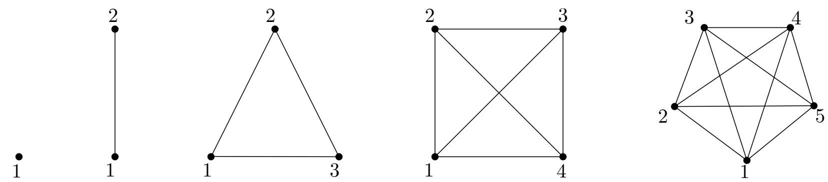


(a) The graphs $\mathbf{P}_1^{(2)}, \ldots, \mathbf{P}_5^{(2)}$



(b) The graphs $\mathbf{Q}_1^{(2)}, \ldots, \mathbf{Q}_5^{(2)}$

FIGURE 1. Some powers of paths and cycles.

that the distance between any two 1's of the string is greater than h. Following [MS02] (see also [Kla11]), a *Fibonacci string of order* n is a binary strings of length n without (two) consecutive 1's. Recalling that the Hamming distance between two binary strings α and β is the number $H(\alpha, \beta)$ of bits where α and β differ, we can define the *Fibonacci cube of order* n, denoted Γ_n, as the graph (V, E), where V is the set of all Fibonacci strings of order n and, for all $\alpha, \beta \in V$, $(\alpha, \beta) \in E$ if and only if $H(\alpha, \beta) = 1$. One can observe that for $h = 1$ the binary strings associated with independent subsets of $\mathbf{P}_n^{(h)}$ are *Fibonacci strings of order* n, and the Hasse diagram of the set of all such strings ordered bitwise is Γ_n. Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [Hsu93], and their combinatorial structure has been further investigated, *e.g.* in [KP07, MS02]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, *e.g.*, [IKR12a, Kla11]).

Remark. Consider the *generalized Fibonacci cubes* described in [IKR12a], *i.e.*, the graphs $B_n(\alpha)$ obtained from the n-cube B_n of all binary strings of length n by removing all vertices that contain the binary string α as a substring. In this notation the Fibonacci cube is $B_n(11)$. It is not difficult to see that $\mathbf{H}_n^{(h)}$ cannot be expressed, in general, in terms of $B_n(\alpha)$. Instead we have:

$$\mathbf{H}_n^{(2)} = B_n(11) \cap B_n(101)\,,\ \mathbf{H}_n^{(3)} = B_n(11) \cap B_n(101) \cap B_n(1001)\,,\ \ldots\,,$$

where $B_n(\alpha) \cap B_n(\beta)$ is the subgraph of B_n obtaining by removing all strings that contain either α or β.

A similar argument can be carried out in the case of cycles. Indeed, every independent subset S of $\mathbf{Q}_n^{(h)}$ can be represented by a circular binary string (*i.e.*, a sequence of 0's and 1's with the first and last bits considered to be adjacent) $b_1 b_2 \cdots b_n$, where, for $i = 1, \ldots, n$, $b_i = 1$ if and only if $v_i \in S$. Thus, each independent subset of $\mathbf{Q}_n^{(h)}$ is associated with a circular binary string of length n such that the distance between any two 1's of the string is greater than h. A *Lucas cube of order* n, denoted Λ_n, is defined as the graph whose vertices are the binary strings of length n without either two consecutive 1's or a 1 in the first and in the last position, and in which the vertices are adjacent when their Hamming distance is exactly 1 (see [MPCZS01]). For $h = 1$ the Hasse diagram of the set of all circular binary strings associated with independent subsets of $\mathbf{Q}_n^{(h)}$ ordered bitwise is Λ_n. A generalization of the notion of Lucas cubes has been proposed in [IKR12b].

Remark. Consider the *generalized Lucas cubes* described in [IKR12b], that is, the graphs $B_n(\widehat{\alpha})$ obtained from the n-cube B_n of all binary strings of length n by removing all vertices that have a *circular containing* α as a substring (*i.e.*, such that α is contained in the circular binary strings obtained by connecting first and last bits of the string). In this notation the Lucas cube is $B_n(\widehat{11})$. It is not difficult to see that $\mathbf{M}_n^{(h)}$ cannot be expressed, in general, in terms of $B_n(\widehat{\alpha})$. Instead we have:

$$\mathbf{M}_n^{(2)} = B_n(\widehat{11}) \cap B_n(\widehat{101})\,,\ \mathbf{M}_n^{(3)} = B_n(\widehat{11}) \cap B_n(\widehat{101}) \cap B_n(\widehat{1001})\,,\ \ldots$$

As far as we now, our $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ are new generalizations of Fibonacci and Lucas cubes, respectively.

In the first part of the paper we evaluate $p_n^{(h)}$, *i.e.*, the number of *independent subsets* of $\mathbf{P}_n^{(h)}$, and $H_n^{(h)}$, *i.e.*, the number of edges of $\mathbf{H}_n^{(h)}$. Our main result (Theorem 3.4) is that, for $n, h \geq 0$, the sequence $H_n^{(h)}$ is obtained by convolving the sequence $\underbrace{1, \ldots, 1}_{h}, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots$ with itself.

In the second part of the paper we derive similar results for $q_n^{(h)}$, *i.e.*, the number of *independent subsets* of $\mathbf{Q}_n^{(h)}$, and $M_n^{(h)}$, *i.e.*, the number of edges of $\mathbf{M}_n^{(h)}$.

2. The independent subsets of powers of paths

For $n, h, k \geq 0$, we denote by $p_{n,k}^{(h)}$ the number of independent k-subsets of $\mathbf{P}_n^{(h)}$.

Remark. For $h = 1$, $p_{n,k}^{(h)}$ counts the number of binary strings $\alpha \in \Gamma_n$ such that $H(\alpha, 00\cdots0) = k$.

Lemma 2.1. *For $n, h, k \geq 0$,*

$$p_{n,k}^{(h)} = \binom{n - hk + h}{k}\,.$$

This result is Theorem 1 of [Hog70]. Below we write down a different proof.

Proof. By Definition 1.2, any two elements v_i, v_j of an independent subset of $\mathbf{P}_n^{(h)}$ must satisfy $|j - i| > h$. It is straightforward to check that whenever $n - hk - h < 0$, $p_{n,k}^{(h)} = 0 = \binom{n-hk+h}{k}$. It is also immediate to see that when $n = h = 0$ our lemma holds true.

Suppose now $n - hk - h \geq 0$. We can complete the proof of our lemma by establishing a bijection between independent k-subset of $\mathbf{P}_n^{(h)}$ and k-subsets of a set with $(n - hk + h)$ elements. Let $\mathscr{K}$ be the set of all k-subsets of a set $B = \{b_1, b_2, \ldots, b_{n-hk+h}\}$, and $\mathcal{I}_k$ the set of all independent k-subsets of $\mathbf{P}_n^{(h)}$. Consider the map $f : \mathscr{K} \to \mathcal{I}_k$ such that, for any $S = \{b_{i_1}, b_{i_2}, \ldots, b_{i_k}\} \in \mathscr{K}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n - hk + h$,

$$f(\{b_{i_1}, b_{i_2}, \ldots, b_{i_j}, \ldots, b_{i_k}\}) = \{v_{i_1}, v_{i_2+h}, \ldots, v_{i_j+(j-1)h}, \ldots, v_{i_k+(k-1)h}\}\,.$$

Claim 1. The map f associates with each k-subset $S = \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\} \in \mathscr{K}$ an independent k-subset of $\mathbf{P}_n^{(h)}$.

To see this we first remark that $f(S)$ is a k-subset of $V(\mathbf{P}_n^{(h)})$. Furthermore, for each pair $b_{i_j}, b_{i_{j+t}} \in S$, with $t > 0$, we have

$$i_{j+t} + (j+t-1)h - (i_j + (j-1)h) = i_{j+t} - i_j + th > h\,.$$

Hence, by Definition 1.1, $(f(b_{i_j}), f(b_{i_{j+t}})) = (v_{i_j+(j-1)h}, v_{i_{j+t}+(j+t-1)h}) \notin E(\mathbf{P}_n^{(h)})$. Thus, $f(S)$ is an independent subset of $\mathbf{P}_n^{(h)}$.

Claim 2. The map f is bijective.

It is easy to see that f is injective. Then, we consider the map $f^{-1} : \mathcal{I}_k \to \mathscr{K}$ such that, for any $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \in \mathcal{I}$, with $1 \le i_1 < i_2 < \dots < i_k \le n$,

$$f^{-1}(\{v_{i_1}, v_{i_2}, \dots, v_{i_j}, \dots, v_{i_k}\}) = \{b_{i_1}, b_{i_2-h}, \dots, b_{i_j-(j-1)h}, \dots, b_{i_k-(k-1)h}\}\,.$$

Following the same steps as for f, one checks that f^{-1} is injective. Thus, f is surjective.

By Claims 1 and 2 we have established a bijection between independent k-subsets of $\mathbf{P}_n^{(h)}$ and k-subsets of a set with $(n - hk + h) \ge 0$ elements. The lemma is proved. □

The coefficients $p_{n,k}^{(h)}$ also enjoy the following property: $p_{n,k}^{(h)} = p_{n-k+1,k}^{(h-1)}$.

For $n, h \ge 0$, the number of all independent subsets of $\mathbf{P}_n^{(h)}$ is

$$p_n^{(h)} = \sum_{k \ge 0} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n - hk + h}{k}\,.$$

Remark. Denote by F_n the n^{th} element of the Fibonacci sequence $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$ is the number of elements of the Fibonacci cube of order n.

The following, simple fact is crucial for our work.

Lemma 2.2. *For $n, h \ge 0$,*

$$p_n^{(h)} = \begin{cases} n+1 & \text{if } \; n \le h+1\,, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } \; n > h+1\,. \end{cases}$$

A proof of this Lemma can be also obtained using the first part of [Hog70, Proof of Theorem 1].

Proof. For $n \le h+1$, by Definition 1.2, the independent subsets of $\mathbf{P}_n^{(h)}$ have no more than 1 element. Thus, there are $n+1$ independent subsets of $\mathbf{P}_n^{(h)}$.

Consider the case $n > h+1$. Let $\mathcal{I}$ be the set of all independent subsets of $\mathbf{P}_n^{(h)}$, let $\mathcal{I}_{in}$ be the set of the independent subsets of $\mathbf{P}_n^{(h)}$ that contain v_n, and let $\mathcal{I}_{out} = \mathcal{I} \backslash \mathcal{I}_{in}$. The elements of $\mathcal{I}_{out}$ are in one-to-one correspondence with the $p_{n-1}^{(h)}$ independent subsets of $\mathbf{P}_{n-1}^{(h)}$, and those of $\mathcal{I}_{in}$ are in one-to-one correspondence with the $p_{n-h-1}^{(h)}$ independent subsets of $\mathbf{P}_{n-h-1}^{(h)}$. □

3. The poset of independent subsets of powers of paths

Figure 2 shows a few Hasse diagrams $\mathbf{H}_n^{(h)}$. Notice that, as stated in the introduction, for each n, $\mathbf{H}_n^{(1)}$ is the Fibonacci cube Γ_n.

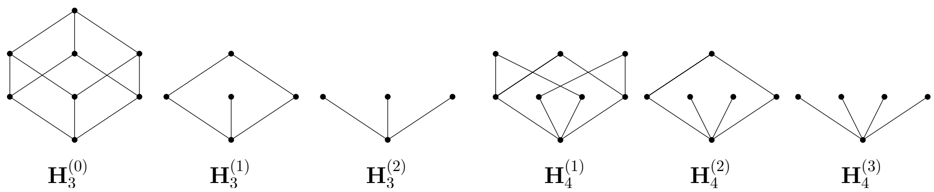


Figure 2. Some $\mathbf{H}_n^{(h)}$.

Let $H_n^{(h)}$ be the number of edges of $\mathbf{H}_n^{(h)}$. Noting that in $\mathbf{H}_n^{(h)}$ each non-empty independent k-subset covers exactly k independent $(k-1)$-subsets, we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1)\rceil} k p_{n,k}^{(h)} = \sum_{k=1}^{\lceil n/(h+1)\rceil} k \binom{n-hk+h}{k} . \tag{1}$$

Remark. For $h = 1$, $H_n^{(h)}$ counts the number of edges of Γ_n.

Let now $T_{k,i}^{(n,h)}$ be the number of independent k-subsets of $\mathbf{P}_n^{(h)}$ containing the vertex v_i, and let, for $h, k \geq 0$, $n \in \mathbb{Z}$, $\bar{p}_{n,k}^{(h)} = \begin{cases} p_{0,k}^{(h)} & \text{if } n < 0\,, \\ p_{n,k}^{(h)} & \text{if } n \geq 0\,. \end{cases}$

Lemma 3.1. *For $n, h, k \geq 0$, and $1 \leq i \leq n$,*

$$T_{k,i}^{(n,h)} = \sum_{r=0}^{k-1} \bar{p}_{i-h-1,r}^{(h)}\ \bar{p}_{n-i-h,k-1-r}^{(h)} .$$

Proof. No independent subset of $\mathbf{P}_n^{(h)}$ containing v_i contains any of the elements $v_{i-h}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+h}$. Let r and s be non-negative integers whose sum is $k-1$. Each independent k-subset of $\mathbf{P}_n^{(h)}$ containing v_i can be obtained by adding v_i to a $(k-1)$-subset $R \cup S$ such that
(a) $R \subseteq \{v_1, \ldots, v_{i-h-1}\}$ is an independent r-subset of $\mathbf{P}_n^{(h)}$;
(b) $S \subseteq \{v_{i+h+1}, \ldots, v_n\}$ is an independent s-subset of $\mathbf{P}_n^{(h)}$.

Viceversa, one can obtain each of this pairs of subsets by removing v_i from an independent k-subset of $\mathbf{P}_n^{(h)}$ containing v_i. Thus, $T_{k,i}^{(n,h)}$ is obtained by counting independently the subsets of type (a) and (b). Noting that the subsets of type (b) are in bijection with the independent s-subsets of $\mathbf{P}_{n-i-h}^{(h)}$, the lemma is proved. □

Remark. $T_{k,i}^{(n,1)}$ counts the number of strings $\alpha = b_1 b_2 \cdots b_n \in \Gamma_n$ such that: (i) $H(\alpha, 00\cdots 0) = k$, and (ii) $b_i = 1$.

In order to obtain our main result, we prepare a lemma.

Lemma 3.2. *For positive* n,

$$\sum_{k=1}^{\lceil n/(h+1)\rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = H_n^{(h)} .$$

Proof. The inner sum counts the number of k-subsets exactly k times, one for each element of the subset. That is, $\sum_{i=1}^{n} T_{k,i}^{(n,h)} = kp_{n,k}^{(h)}$. The lemma follows directly from Equation (1). □

Next we introduce a family of Fibonacci-like sequences.

Definition 3.3. For $h \geq 0$, and $n \geq 1$, we define the *h-Fibonacci sequence* $\mathcal{F}^{(h)} = \{F_n^{(h)}\}_{n\geq 1}$ whose elements are

$$F_n^{(h)} = \begin{cases} 1 & \text{if } \; n \leq h+1 \,, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } \; n > h+1. \end{cases}$$

From Lemma 2.2, and setting for $h \geq 0$, and $n \in \mathbb{Z}$, $\bar{p}_n^{(h)} = \begin{cases} p_0^{(h)} & \text{if } \; n < 0 \,, \\ p_n^{(h)} & \text{if } \; n \geq 0 \,, \end{cases}$ we have that,

$$F_i^{(h)} = \bar{p}_{i-h-1}^{(h)} \,, \quad \text{for each } \; i \geq 1 \,. \tag{2}$$

Thus, our Fibonacci-like sequences are obtained by adding a prefix of h ones to the sequence $p_0^{(h)}, p_1^{(h)}, \ldots$. Therefore, we have:

- $\mathcal{F}^{(0)} = 1, 2, 4, \ldots, 2^n, \ldots$;
- $\mathcal{F}^{(1)}$ is the Fibonacci sequence;
- more generally, $\mathcal{F}^{(h)} = \underbrace{1, \ldots, 1}_{h}, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots$.

In the following, we use the discrete convolution operation $*$, as follows.

$$\left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n) \doteq \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)} \tag{3}$$

Theorem 3.4. *For* $n, h \geq 0$, *the following holds.*

$$H_n^{(h)} = \left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)}\right)(n) .$$

Proof. The sum $\sum_{k=1}^{\lceil n/(h+1)\rceil} T_{k,i}^{(n,h)}$ counts the number of independent subsets of $\mathbf{P}_n^{(k)}$ containing v_i. We can also obtain such a value by counting the independent subsets of both $\{v_1, \ldots, v_{i-h-1}\}$, and $\{v_{i+h+1}, \ldots, v_n\}$. Thus, we have:

$$\sum_{k=1}^{\lceil n/(h+1)\rceil} T_{k,i}^{(n,h)} = \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} .$$

Using Lemma 3.2 we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1)\rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \sum_{k=1}^{\lceil n/(h+1)\rceil} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.$$

By Equation (2) we have $\sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)}$. By (3), the theorem is proved. □

Remark. For $h = 1$, we obtain the number of edges of Γ_n by using Fibonacci numbers:

$$H_n^{(h)} = \sum_{i=1}^{n} F_i F_{n-i+1} \,.$$

The latter result is [Kla05, Proposition 3].

4. The independent subsets of powers of cycles

For $n, h, k \geq 0$, we denote by $q_{n,k}^{(h)}$ the number of independent k-subsets of $\mathbf{Q}_n^{(h)}$.

Remark. For $h = 1$, $n > 1$, $q_{n,k}^{(h)}$ counts the number of binary strings $\alpha \in \Lambda_n$ such that $H(\alpha, 00\cdots0) = k$.

Lemma 4.1. *For $n, h \geq 0$, and $k > 1$,*

$$q_{n,k}^{(h)} = \frac{n}{k}\binom{n-hk-1}{k-1} \,.$$

Moreover, $q_{n,0}^{(h)} = 1$, and $q_{n,1}^{(h)} = n$, for each $n, h \geq 0$.

Proof. Fix an element $v_i \in V(\mathbf{Q}_n^{(h)})$, and let $n > 2h$. Any independent subset of $\mathbf{Q}_n^{(h)}$ containing v_i does not contain the h elements preceding v_i and the h elements following v_i. Thus, the number of independent k-subsets of $\mathbf{Q}_n^{(h)}$ containing v_i equals

$$p_{n-2h-1,k-1}^{(h)} = \binom{n-hk-1}{k-1} \,.$$

The total number of independent k-subsets of $\mathbf{Q}_n^{(h)}$ is obtained by multiplying $p_{n-2h-1,k-1}^{(h)}$ by n, then dividing it by k (each subset is counted k times by the previous proceeding). The case $n \leq 2h$, as well as the cases $k = 0, 1$, can be easily verified. □

For $n, h \geq 0$, the number of all independent subsets of $\mathbf{Q}_n^{(h)}$ is

$$q_n^{(h)} = \sum_{k\geq 0} q_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1)\rceil} q_{n,k}^{(h)} \,, \tag{4}$$

Remark. Denote by L_n the n^{th} element of the Lucas sequence $L_1 = 1$, $L_2 = 3$, and $L_i = L_{i-1} + L_{i-2}$, for $i > 2$. Then, for $n > 1$, $q_n^{(1)} = L_n$ is the number of elements of the Lucas cube of order n.

The coefficients $q_n^{(h)}$ satisfy a recursion that closely resemble that of Lemma 2.2.

Lemma 4.2. *For $n, h \geq 0$,*

$$q_n^{(h)} = \begin{cases} n+1 & \textit{if} \;\; n \leq 2h+1 \,, \\ q_{n-1}^{(h)} + q_{n-h-1}^{(h)} & \textit{if} \;\; n > 2h+1. \end{cases} \tag{5}$$

Proof. The case $n \leq 2h+1$ can be easily checked. Let $n > 3h+2$, and let $\mathcal{I}$ be the set of the independent subsets of $\mathbf{Q}_n^{(h)}$. Let $\mathcal{I}_{in}$ be the subset of these subsets that (i) do not contain v_n, and that (ii) contain no one of the following pairs: $(v_1, v_{n-h}), (v_2, v_{n-h+1}), \ldots, (v_h, v_{n-1})$. Furthermore let $\mathcal{I}_{out}$ be the subset of the remaining independent subsets of $\mathbf{Q}_n^{(h)}$.

It is easy to see that the elements of $\mathcal{I}_{in}$ are exactly the independent subsets of $\mathbf{Q}^{(h)}_{n-1}$. Indeed, v_n is not a vertex of $\mathbf{Q}^{(h)}_{n-1}$ and the vertices of pairs (v_1, v_{n-h}), (v_2, v_{n-h+1}), $\ldots$, (v_h, v_{n-1}) are connected in $\mathbf{Q}^{(h)}_{n-1}$. On the other hand, to show that

$$|\mathcal{I}_{out}| = q^{(h)}_{n-h-1}$$

we argue as follows. First we recall (see the proof of Lemma 4.1) that the number of independent k-subsets of $\mathbf{Q}^{(h)}_n$ that contain v_n is $p^{(h)}_{n-2h-1,k-1}$. Secondly we obtain that the number of independent k-subsets of $\mathbf{Q}^{(h)}_n$ containing one of the pairs (v_1, v_{n-h}), (v_2, v_{n-h+1}), $\ldots$, (v_h, v_{n-1}) is $hp^{(h)}_{n-3h-2,k-2}$. To see this, consider the pair (v_1, v_{n-h}). The independent subsets containing such a pair do not contain the h vertices from $v_n - h + 1$ to v_n, do not contain the h vertices from v_2 to v_{h+1}, and do not contain the h vertices from v_{n-2h} to v_{n-h-1}. Thus, the removal of such vertices and of the vertices v_1 and v_{n-h} turns $\mathbf{Q}^{(h)}_n$ into $\mathbf{P}^{(h)}_{n-3h-2}$. Hence we can obtain all the independent k-subsets of $\mathbf{Q}^{(h)}_n$ that contain the pair (v_1, v_{n-h}) by simply adding these two vertices to one of the $p^{(h)}_{n-3h-2,k-2}$ independent $k-2$-subsets of $\mathbf{P}^{(h)}_{n-3h-2}$. Same reasoning can be carried out for any other one of the pairs: (v_2, v_{n-h+1}), $\ldots$, (v_h, v_{n-1}).

Using Lemmas 2.1 and 4.1 one can easily derive that

$$p^{(h)}_{n-2h-1,k-1} + hp^{(h)}_{n-3h-2,k-2} = q^{(h)}_{n-h-1,k-1}\,.$$

Hence, we derive the size of $\mathcal{I}_{out}$:

$$|\mathcal{I}_{out}| = q^{(h)}_{n-h-1} = \sum_{k\geq 1} p^{(h)}_{n-2h-1,k-1} \; + h \sum_{k\geq 2} p^{(h)}_{n-3h-2,k-2}\,.$$

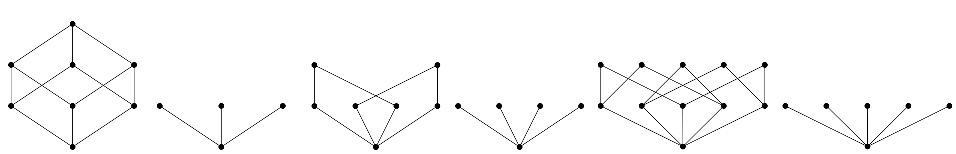
Summing up we have shown that $|\mathcal{I}| = |\mathcal{I}_{in}| + |\mathcal{I}_{out}|$, that is

$$q^{(h)}_n = q^{(h)}_{n-1} + q^{(h)}_{n-h-1}\,.$$

The proof of the case $2h+1 < n \leq 3h+2$ is obtained in a similar way, observing that $|\mathcal{I}_{out}| = n - h$, and that $n - h - 1 \leq 2h + 1$. □

5. The poset of independent subsets of powers of cycles

Figure 3 shows a few Hasse diagrams $\mathbf{M}^{(h)}_n$. Notice that, as stated in the introduction, for each n, $\mathbf{M}^{(1)}_n$ is the Lucas cube Λ_n.



$\mathbf{L}^{(0)}_3$ $\mathbf{L}^{(1)}_3$ $\mathbf{L}^{(1)}_4$ $\mathbf{L}^{(2)}_4$ $\mathbf{L}^{(1)}_5$ $\mathbf{L}^{(2)}_5$

Figure 3. Some $\mathbf{M}^{(h)}_n$.

Let $M_n^{(h)}$ be the number of edges of $\mathbf{M}_n^{(h)}$. As done in Section 3 for the case of paths, we immediately provide a formula for $M_n^{(h)}$:

$$M_n^{(h)} = \sum_{k=0}^{\lceil n/h+1 \rceil} k q_{n,k}^{(h)} = n \sum_{k=0}^{\lceil n/h+1 \rceil} \binom{n-hk-1}{k-1} . \tag{6}$$

Remark. For $h = 1$, $n > 1$, $M_n^{(h)}$ counts the number of edges of Λ_n. As shown in [MPCZS01, Proposition 4(ii)], $M_n^{(h)} = nF_{n-1}$.

As shown in the proof of Lemma 4.1, the value

$$p_{n-2h-1,k-1}^{(h)} = \binom{n-hk-1}{k-1}$$

is the analogue of the coefficient $T_{k,i}^{(n,h)}$: in the case of cycles we have no dependencies on i, because each choice of vertex is equivalent. We can obtain $M_n^{(h)}$ in terms of a fibonacci-like sequence, as follows.

Proposition 5.1. *For $n > h \geq 0$, the following holds.*

$$M_n^{(h)} = nF_{n-h}^{(h)} .$$

Proof. Using Equation (2) we obtain:

$$M_n^{(h)} = n \sum_{k=1}^{\lceil n/(h+1) \rceil} \bar{p}_{n-2h-1,k-1}^{(h)} = n\bar{p}_{n-2h-1}^{(h)} = nF_{n-h}^{(h)} .$$

□

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