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A THEORY OF PARTITIONS OF PARTIALLY ORDERED SETS

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Introduction

A partition of a set A is a set of nonempty pairwise disjoint subsets of A whose union is A. An equivalent definition of a partition can be given using functions. In these terms, a partition of a set is the set of fibres of a surjective function.

The latter definition allows us to introduce the central topic of this thesis: a notion of partition for partially ordered sets. Analyzing the category `Poset` of partially ordered sets, or posets, and order-preserving maps, we see that two kinds of surjections have to be taken into account: order-preserving surjections, and the stronger regular surjections. Therefore, we must deal with two different classes of partitions, namely, *monotone* and *regular* partitions of a poset. These two notions form the basis for a theory of partitions of posets. In analogy with the set-theoretic case, our first step is to obtain characterizations of monotone and regular partitions which do not mention order-preserving maps. Throughout, we restrict attention to the case of finite posets.

Our analysis of the category `Poset` is presented in Chapter 2. Here, we introduce the notions of epimorphism and regular epimorphism, the classes of order-preserving surjections needed to define partitions of posets. The use of precisely these two kinds of maps is justified by the existence of two dual factorization systems, that we illustrate in the last section of the chapter.

The third chapter is the core of the thesis. We introduce the definitions of monotone and regular partitions of a poset, and obtain their characterizations. At the end of the chapter, we present such partitions of a poset as extensions of some specific partitions of the underlying set of the poset. Such extensions are obtained by endowing the underlying partitions with an appropriate order. We introduce a necessary and sufficient condition for a partition to be extended, and we prove that if our condition is satisfied, the extension of a partition of the underlying set of a poset P to a regular partition of P is unique.

In Chapter 4, we study the collection of all monotone and regular partitions of a poset. We endow these classes with a lattice structure, obtaining the monotone and regular partition lattices of a poset. First, a bijection between monotone and regular partitions and certain classes of quasiorders is established. This result generalizes the usual correspondence between partitions and equivalence relations. Then, we describe the monotone and regular partition lattices in terms of quasiorders. We obtain a description of the lattice operations in terms of operations on quasiorders, and also more directly in terms of partitions. Finally, we investigate these lattices and some of their properties. We obtain in particular that the lattice of regular partitions of a poset P can be embedded both as a join-subsemilattice in the partition lattice of the underlying set of P, and as a meet-subsemilattice in the monotone partition lattice of P. We also prove that the regular partition lattice is ranked, while the monotone partition lattice, in general, is not.

In Chapter 5, we present the well-known Birkhoff duality between Poset and the category of bounded distributive lattices and their $\{\mathbf{0},\mathbf{1}\}$-preserving lattice homomorphisms. We thus investigate the duals of monotone and regular partitions via this duality. The dual notions to monotone and regular partitions are sublattices and regular sublattices, respectively. We characterize regular sublattices as sublattices with an additional algebraic property.

Finally, the last chapter is devoted to a first discussion of enumeration problems within the theory of partitions of finite posets. In particular, we study the monotone and regular partition lattices of chains and antichains, and count the number of their elements. Building on this, we provide tight bounds for the number of monotone and regular partitions of any poset. Then, by way of illustration, we provide a formula to compute the number of regular partitions for a specific family of posets. In the last two sections we deal with the enumeration of order-preserving maps. In particular, the last section is devoted to Möbius inversion. Here, we present the well-known Möbius inversion theorem, and we give a numerical example of a Möbius inversion on the lattice of regular partitions of a given poset.

Chapter 1

Basic notions

Pigmaei gigantum humeris impositi plusquam ipsi gigantes vident.

Isaac Newton

1.1 Partially ordered sets

The principal notion which appear in this thesis is that of *partially ordered set*. Consider a set P. A *partial order* on P is a binary relation $\leqslant$ on P such that, for all $x,y \in P$,

- $x \leqslant x$ (*reflexivity*),
- $x \leqslant y$ and $y \leqslant x$ imply $x = y$ (*antisymmetry*),
- $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$ (*transitivity*).

The fact that $x \leqslant y$ can also be expressed by $(x,y) \in \leqslant$. A set P equipped with a partial order $\leqslant$ is called a *partially ordered set*, or *poset*, for short. A poset is usually denoted simply by specifying the *underlying set* P, except when it is necessary to specify the order. In this case, we use the notation $(P,\leqslant)$. We sometimes write $y \geqslant x$, instead of $x \leqslant y$, with the same meaning. We write $x \nleqslant y$ to mean that $x \leqslant y$ does not hold. If $x \nleqslant y$ and $y \nleqslant x$ we write $x \parallel y$, and say that x and y are *incomparable*. We also use the symbol $<$ to denote a *strict inequality*, that is $x < y$ if and only if $x \leqslant y$ and $x \neq y$.

A *finite poset* is a poset whose underlying set is finite.

Let P be a poset. If for all $x,y \in P$, either $x \leqslant y$ or $y \leqslant x$, then P is a *chain*. Alternative names for a chain are *linearly ordered set* and *totally ordered set*. The poset P is an *antichain* if for $x,y \in P$, $x \leqslant y$ only if $x = y$.

Another important notion is that of *covering relation*. Let P be a poset, and let $x,y \in P$. We say that x *is covered by* y, or that y *covers* x, and write $x \lhd y$, if $x < y$ and there is no element $z \in P$ such that $x < z < y$. The covering relation is particulary useful when drawing posets. In fact, we can represent a poset P by a configuration of points (the elements of P) and interconnecting lines indicating the covering relation. For example, Figure 1.2 depicts a poset P whose underlying set is $\{a,b,c,d\}$, endowed with the partial order $\leqslant$ such that $a < c$, $a < d$, $b < c$, $b < d$, and no other pairs of distinct elements are comparable. Such a

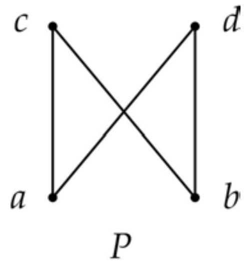


Figure 1.1: *Hasse Diagram.*

representation of a poset is called its *Hasse diagram*. Following tradition, we do not give a formal definition of Hasse diagram.

Let $(P, \leqslant)$ be a poset and $Q \subseteq P$. The *downset* of Q, written $\downarrow Q$, is defined by

$$\downarrow Q = \{x \in P \mid x \leqslant y, \text{ for some } y \in Q\}.$$

We write $\downarrow x$ for $\downarrow \{x\}$. A partially ordered set $(P, \leqslant)$ is a *forest* if for all $x \in P$ the downset $\downarrow x$ is a chain. An element $m \in P$ is *minimal* if $x \nleqslant m$ for all $x \in P$, and is the *bottom* or *minimum* of P if $m \leqslant x$ for all $x \in P$. A *tree* is a forest with a bottom element, called the *root* of the tree. A *subforest* of a forest P is the downset of some $Q \subseteq P$.

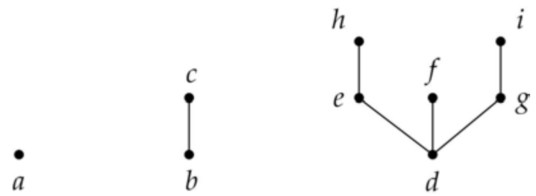


Figure 1.2: *A forest.*

In this thesis, and in particular in Chapters 4 and 6, we make use of some specific combinatorial properties of posets. Although many of these properties will be introduced when needed, it is convenient to introduce here the notion of *ranked poset*.

Definition 1.1. A poset P is said to be *ranked* if there exists a *rank function* $r : P \to \{0, 1, \ldots, n\}$ such that $r(x) = 0$ if x is a minimal element of P, and $r(y) = r(x) + 1$ if y covers x in P.

If P is a ranked poset, the number of elements of P of rank k is denoted W_k and is called the k-th *Whitney number of P of the second kind*.

Example 1.1. Figure 1.3 shows the Hasse diagram of a ranked poset. Its Whitney numbers of the second kind are

$$W_0 = 1\,, \;\; W_1 = 6\,, \;\; W_2 = 7\,, \;\; W_3 = 1\,.$$

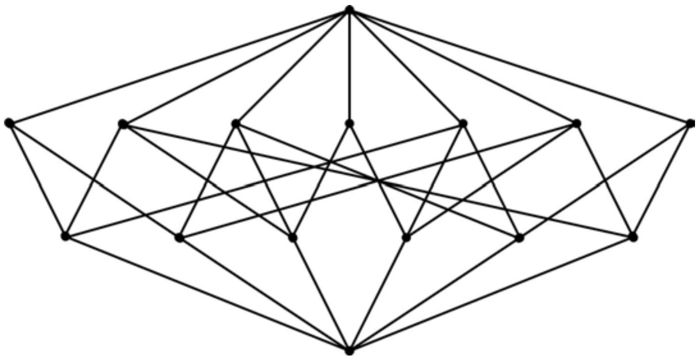


Figure 1.3: *A ranked poset.*

1.2 Order-preserving maps

The following notions play a fundamental role throughout the thesis, and in particular in Chapter 2.

Let P and Q be posets. A map $f : P \to Q$ is said to be *order-preserving*, or *monotone*, if for any $x, y \in P$, $x \leqslant y$ in P implies $f(x) \leqslant f(y)$ in Q.

A map $g : P \to Q$ is said to be an *order-embedding* whenever for any $x, y \in P$, $x \leqslant y$ in P if and only if $g(x) \leqslant g(y)$ in Q. When $g : P \to Q$ is an order-embedding we shall write $g : P \hookrightarrow Q$.

A map $h : P \to Q$ is said to be an *order-isomorphism*, if h is an order-embedding, and it is surjective on the underlying sets. When there exists an order-isomorphism from P to Q, we say that P and Q are *order-isomorphic* and write $P \cong Q$.

We also use the notions of order-preserving injections (surjections), that is order-preserving maps which are injective (surjective) on the underlying sets.

Example 1.2. Figure 1.4 shows an order-preserving map. Figure 1.5 shows an order-embedding which is not an isomorphism. Figure 1.6 shows an order-isomorphism. Finally, Figure 1.7 shows an order-preserving injection which is not an embedding.

1.3 Notions on lattice theory

Let P be a poset, and let $S \subseteq P$. An element $x \in P$ is an *upper bound* of S if $s \leqslant x$ for all $s \in S$. An element $x \in P$ is the *least upper bound* of S, or *supremum* of S, written $\sup S$, if x is an upper bound of S, and $s \leqslant y$ for every $s \in S$ implies $x \leqslant y$. Similarly, we define a *lower bound* of S as an element $x \in P$ such that $x \leqslant s$ for all $s \in S$. An element $x \in P$ is the *greatest lower bound* of S, or *infimum* of S, written $\inf S$, if x is a lower bound of S, and $y \leqslant s$ for every $s \in S$ implies $y \leqslant x$.

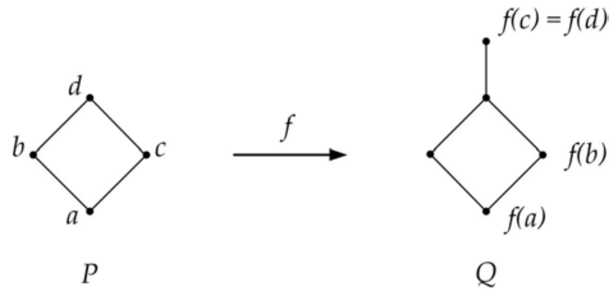


Figure 1.4: *An order-preserving map.*

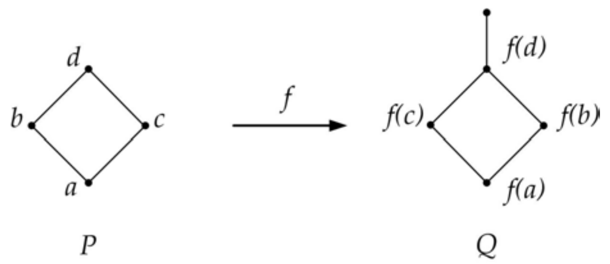


Figure 1.5: *An order-embedding.*

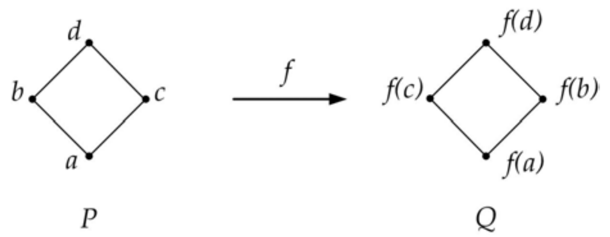


Figure 1.6: *An order-isomorphism.*

Definition 1.2. A poset L is a *lattice* if either $L = \emptyset$, or for every $x, y \in L$ both $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in L.

We write $x \vee y$ in place of $\sup\{x, y\}$, and $x \wedge y$ in place of $\inf\{x, y\}$. We thus have $x \leqslant y$ if and only if $x = x \wedge y$. With the use of $\wedge$ (*meet*) and $\vee$ (*join*) we can regard lattices as algebraic structures.

Definition 1.3. An algebraic structure $(L, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations, is a *lattice* if it satisfies the following identities.

(L1) $x \vee y = y \vee x, \; x \wedge y = y \wedge x$. (Commutative laws)

(L2) $x \vee (y \vee z) = (x \vee y) \vee z, \; x \wedge (y \wedge z) = (x \wedge y) \wedge z$. (Associative laws)

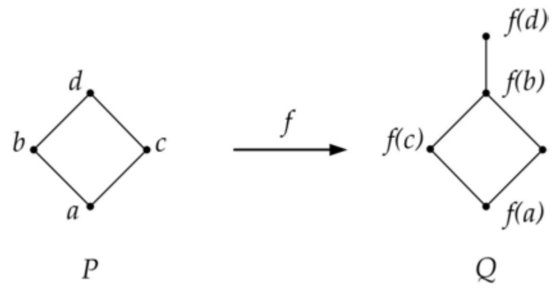


Figure 1.7: *An order-preserving injection.*

(L3) $x \vee x = x$, $x \wedge x = x$. (Idempotent laws)

(L4) $x = x \vee (x \wedge y)$, $x = x \wedge (x \vee y)$. (Absorption laws)

It is not difficult to verify that the two definitions of lattice are equivalent, in the sense that if L is a lattice by one of the two definitions, then it is a lattice by the other.

A lattice L is said to be *complete* if for every subset S of L, both $\sup S$ and $\inf S$ exist in L. One can easily verify that every finite lattice is complete.

> *In this thesis, we are concerned with finite posets and lattices. Thus, from now on 'poset' means 'finite poset', and 'lattice' means 'finite lattice', unless otherwise stated.*

A lattice always has a bottom element, denoted $\perp$, or $\mathbf{0}$, and a top element, denoted $\top$, or $\mathbf{1}$. The elements which cover the bottom are called *atoms*. Sometimes the term *coatoms* is also used to indicate the elements of a lattice which are covered by the top.

Example 1.3. Figure 1.8 show the Hasse diagram of a lattice. The element t is the top, b is the bottom. The atoms are x and y, the coatoms are y and z. The poset in Figure 1.9 is not a lattice, for the supremum of x and y does not exist.

The most thoroughly studied classes of lattices are distributive and modular lattices. The former will have a fundamental role in Chapter 5.

Definition 1.4. A *distributive lattice* is a lattice which satisfies the *distributive laws*

(D1) $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$,

(D2) $x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$.

Definition 1.5. A *modular lattice* is a lattice which satisfies the *modular law*

(M) $x \leqslant y$ implies $x \vee (y \wedge z) = x \wedge (y \vee z)$.

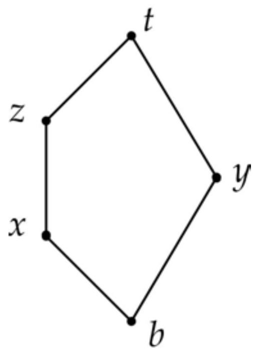


Figure 1.8: *A lattice.*

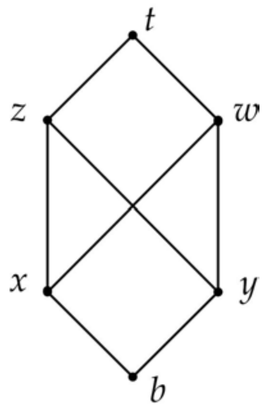


Figure 1.9: *A poset which is not a lattice.*

It is possible to show that every distributive lattice is modular.

A lattice L is called *semimodular* if it satisfies the *upper covering condition*, that is whenever x and y cover $x \wedge y$ in L, then $x \vee y$ covers both x and y. A lattice L is *atomic* if each element of L is a join of atoms. A lattice which is atomic and semimodular is called a *geometric lattice*. An important family of geometric lattices is that of *partition lattices*, which will be introduced in Chapter 4.

Example 1.4. Figure 1.10 shows the Hasse diagram of a geometric lattice. One can check that the depicted lattice is semimodular and atomic. The lattice in Figure 1.8 is a classical example of a non-geometric lattice. In fact, both x and y cover $x \wedge y$, but $x \vee y$ does not cover x.

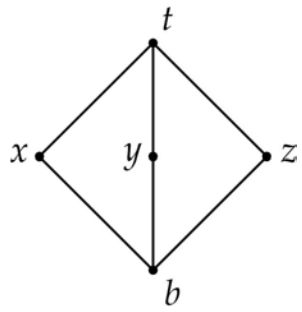


Figure 1.10: *A geometric lattice.*

1.4 Bibliographic notes

Almost all the basic notions presented in this chapter can be found in [DP02]. However, note that while [DP02] require a lattice to be nonempty, we admit empty lattices. We also refer to [Grä98], where many more details on lattices can be found. An introduction to lattices can also be found in every book on universal algebra. We cite, for example, [BS81]. A different, more combinatorial, approach to posets and lattices can be found in [Sta97].

Chapter 2

The category Poset

If the point is sharp, and the arrow is swift, it can pierce through the dust no matter how thick.

Bob Dylan

Note. As specified in the introduction, we always deal with finite objects, although some results can be easily adapted to the infinite case. Thus, when we talk about posets we always mean finite posets, and when we talk about concrete categories whose objects are sets with structure, the underlying sets are always assumed to be finite.

2.1 Basics on categories

In this chapter we give a categorical introduction to the objects of our study: partially ordered sets. The main goal is to give a strong motivation for the central subject of this thesis, that is the study of partitions of partially ordered sets. Essentially, studying a category means studying objects together with morphisms between them. Well-known categories are, for instance, sets and functions, partially ordered sets and order-preserving maps, groups and homomorphisms, topological spaces and continuous functions.

We will focus on the category `Poset` of partially ordered sets and order-preserving maps, but, as much as possible, we will draw a parallel between `Poset` and the category `Set` of sets and functions, trying to point out similarities and differences.

Formally, a category $\mathcal{K}$ is a collection of *objects* of $\mathcal{K}$ together with, for each pair A, B of objects, a (possibly empty) collection $\mathcal{K}(A,B)$ of *morphisms* from A to B. We may write $f : A \to B$ or $A \xrightarrow{f} B$ to indicate that the morphism f is in $\mathcal{K}(A,B)$, and we then refer to A as the *domain* of f and to B as the *codomain* of f. Morphisms are subject to the following conditions.

(1) For any three objects A, B and C of $\mathcal{K}$, there is given a *composition* law
$$\mathcal{K}(A,B) \times \mathcal{K}(B,C) \to \mathcal{K}(A,C) : (A \xrightarrow{f} B, B \xrightarrow{g} C) \mapsto A \xrightarrow{g \circ f} C$$
which satisfies the *associative axiom* that for any objects A, B, C, D of $\mathcal{K}$ and all morphisms f in $\mathcal{K}(A,B)$, g in $\mathcal{K}(B,C)$, h in $\mathcal{K}(C,D)$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(2) For every object A of $\mathcal{K}$, the collection $\mathcal{K}(A,A)$ contains a special morphism id_A, called the *identity* of A, with the property that for every object B of $\mathcal{K}$, and for all f in $\mathcal{K}(A,B)$ and g in $\mathcal{K}(B,A)$ we have $f \circ id_A = f$ and $id_A \circ g = g$.

We often use *commutative diagrams* like the one in Figure 2.1 to represent properties of morphisms in a category, and we say that a diagram *commutes* if we can exchange paths, between two given points, with impunity.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \cdot f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Figure 2.1: *Commutative diagram.*

We have already specified that `Poset` shall have *posets* for objects, and that for each pair of objects A and B, $\texttt{Poset}(A, B)$ shall be the set of order-preserving maps from A to B. Let $f : A \to B$ and $g : B \to C$ be order-preserving maps. Composition is defined as $(g \circ f)(x) = g(f(x))$, and identity is defined as $id_A(x) = x$, for all $x \in A$. Then

$$a \leqslant a' \text{ in } A \Rightarrow f(a) \leqslant f(a') \text{ in } B \Rightarrow g(f(a)) \leqslant g(f(a')) \text{ in } C$$

so that $g \circ f$ is order-preserving. Clearly, id_A is order-preserving. Since composition and identities are defined as in `Set`, we see that `Poset` is indeed a category.

Some morphisms play a distinguished role in categories, as we will soon see for the category `Set`. These fundamental classes of morphism are *monomorphisms*, *epimorphisms* and *isomorphisms*.

Definition 2.1. A morphism $f : A \to B$ is said to be a *monomorphism* provided that for all pairs $h, k : C \to A$ of morphisms such that $f \circ h = f \circ k$, it follows that $h = k$. In other words f is *left-cancellable* with respect to composition.

$$\begin{tikzcd} C \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Figure 2.2: *Monomorphism.*

The concept of category affords an economical and useful duality: each concept is two concepts, and each result is two results. The categorical dual of monomorphism is epimorphism.

Definition 2.2. A morphism $f : A \to B$ is said to be an *epimorphism* provided that for all pairs $h, k : B \to C$ of morphisms such that $h \circ f = k \circ f$, it follows that $h = k$. In other words f is *right-cancellable* with respect to composition.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & C \end{tikzcd}$$

Figure 2.3: *Epimorphism.*

A function is a monomorphism in `Set` if and only if it is injective. A function is an epimorphism in `Set` if and only if it is surjective. The next proposition gives a characterization of monomorphisms and epimorphisms in `Poset`.

Proposition 2.1. *An order-preserving map is a monomorphism in* `Poset` *if and only if it is injective. An order-preserving map is an epimorphism in* `Poset` *if and only if it is surjective.*

Proof. Let $f : A \to B$ be an order-preserving injection, and consider two order-preserving maps $h, k : C \to A$ such that $f(h(c)) = f(k(c))$ for every $c \in C$. Since f is injective, this implies that $h(c) = k(c)$ for every $c \in C$.

Take now $f : A \to B$ to be mono,[1] and suppose it is not an injection. Let $a_1, a_2 \in A$ be such that $a_1 \neq a_2$ and $f(a_1) = f(a_2)$. Consider a poset C with only two elements $c_1 \parallel c_2$, an order-preserving map $h : C \to A$ defined by $h(c_1) = h(c_2) = a_1$, and an order-preserving map $k : C \to A$ defined by $k(c_1) = a_1$, $k(c_2) = a_2$. We have $f \circ h = f \circ k$, but $h \neq k$.

In a similar way, we can obtain the second statement. □

Definition 2.3. A morphism $f : A \to B$ in a category $\mathcal{K}$ is called an *isomorphism* provided that there exists a morphism $g : B \to A$ with $g \circ f = id_A$ and $f \circ g = id_B$. Such a morphism g is called an *inverse* of f.

In the category `Set` the class of isomorphisms coincides with the class of all bijections between sets. A characterization of isomorphisms in `Poset` is given by the following proposition.

Proposition 2.2. *In* `Poset` *isomorphisms are precisely order-isomorphisms.*

Proof. Let $f : A \to B$ be an order-isomorphism, and consider the function $\hat{f}^{-1} : B \to A$ defined by $\hat{f}^{-1}(b) = a$, with $a \in A$ such that $f(a) = b$ (unique because f is bijective), for all $b \in B$. Since, for each $a_1, a_2 \in A$, $a_1 \leqslant a_2$ if and only if $f(a_1) \leqslant f(a_2)$, $\hat{f}^{-1}$ can be regarded as an order-preserving map $f^{-1} : B \to A$ such that for each $b_1, b_2 \in B$, $b_1 \leqslant b_2$ if and only if $f^{-1}(b_1) \leqslant f^{-1}(b_2)$. Moreover, by construction, $f^{-1}(f(a)) = a$, for all $a \in A$, and $f(f^{-1}(b)) = b$, for all $b \in B$.

Take now $f : A \to B$ to be iso in the category `Poset`. Let $g : B \to A$ be the inverse of f. For $h, k : B \to C$ we note that $k \circ f = h \circ f$ implies $k \circ f \circ g = h \circ f \circ g$. We obtain $h = k$, and thus f is epi. We also note, for $h', k' : C \to A$, that $f \circ k' = f \circ h'$ implies $g \circ f \circ k' = g \circ f \circ h'$. We

[1] We use without distinction "f is mono" and "f is a monomorphism". The same applies to "f is epi" and "f is an epimorphism".

obtain $h' = k'$, and thus f is mono. Finally, since f and g are order-preserving, for $x, y \in A$, we have that $x \leqslant y$ implies $f(x) \leqslant f(y)$, and that $f(x) \leqslant f(y)$ implies $x = g(f(x)) \leqslant g(f(y)) = y$, so f is an order-isomorphism. $\square$

The last concept we introduce in this paragraph is that of *factorization system*.

Definition 2.4. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms of a category $\mathcal{K}$. $\mathcal{K}$ is a *uniquely $(\mathcal{E},\mathcal{M})$-factorizable category* if and only if

- each morphism f has an $(\mathcal{E},\mathcal{M})$-factorization $f = m \circ e$, with m in $\mathcal{M}$ and e in $\mathcal{E}$,
- $(\mathcal{E},\mathcal{M})$-factorizations are *essentially unique*, *i.e.*, whenever $f = m \circ e = m' \circ e'$ for some m, m' in $\mathcal{M}$, e, e' in $\mathcal{E}$, there exists an isomorphism h such that the following diagram commutes.

$$
\begin{tikzcd}
 & B \arrow[dr, "m"] \arrow[dd, dashed, "h"] & \\
A \arrow[ur, "e"] \arrow[dr, "e'"'] & & C \\
 & B' \arrow[ur, "m'"'] &
\end{tikzcd}
$$

Figure 2.4: *Factorization system.*

If $\mathcal{E}$ and $\mathcal{M}$ are closed under composition and contain all isomorphisms we say that $(\mathcal{E},\mathcal{M})$ is a *factorization system* for the category $\mathcal{K}$.

Let epi be the class of all epimorphisms in a given category, and let mono be the class of all monomorphisms. The well-known fact that each function between sets factorizes as a surjection followed by an injection can be reformulated as follows:

(epi,mono) is a factorization system for the category `Set`.

Figure 2.5 shows the factorization $f = m \circ e$ of a morphism $f : A \to B$ as a composition of an epimorphism $e : A \to f(A)$ and a monomorphism $m : f(A) \to B$.

Unfortunately, the category `Poset` is not uniquely (epi,mono) factorizable, as we show in the following example. Thus, in order to describe a factorization system for `Poset` we need to introduce other classes of morphisms.

Example 2.1. Consider the partially ordered sets P, Q, R, S in Figure 2.6, and consider the order-preserving map $f : P \to Q$ such that $f(a) = z$, $f(b) = f(c) = x$. Let $e : P \to R$ be the order-preserving map defined by $e(a) = v$, $e(b) = e(c) = u$, and let $m : R \to Q$ be such

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & f(A) \arrow[ur, "m"'] & \end{tikzcd}$$

Figure 2.5: *Factorization in* `Set`.

that $m(u) = x$, $m(v) = z$. Then, e is epi in the category `Poset`, and m is mono, because of Proposition 2.1. Moreover, $f = m \circ e$. Let now $e' : P \to S$ be the order-preserving map defined by $e'(a) = v'$, $e'(b) = e'(c) = u'$, and $m' : S \to Q$ be such that $m'(u') = x$, $m'(v') = z$. Again, e' is epi and m' is mono in `Poset`, and $f = m' \circ e'$. Since, by Proposition 2.2, there is no isomorphism between R and S the factorizations $f = m \circ e$ and $f = m' \circ e'$ are not essentially unique, and (epi,mono) is not a factorization system for the category `Poset`.

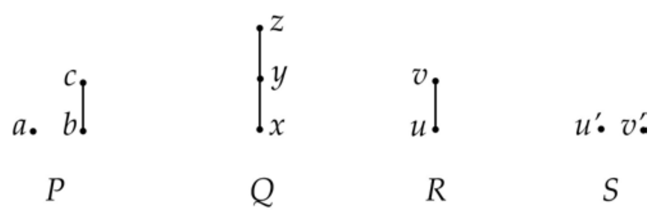


Figure 2.6: *Example 2.1.*

2.2 Regular monomorphisms

Originally it was believed that monomorphisms would constitute the correct categorical abstraction of the notion of "embeddings of substructures" that exists in various contents. However, in many instances the concept of monomorphism is too weak. For example, in `Poset` monomorphisms are just injective order-preserving maps and need not be order-embeddings. We thus introduce a stronger notion that more frequently corresponds to embeddings in categories.

Definition 2.5. A morphism $m : C \to A$ is called a *regular monomorphism* if and only if there exists a pair $f, g : A \to B$ of morphisms such that:

(1) $f \circ m = g \circ m$,

(2) for any morphism $m' : C' \to A$ with $f \circ m' = g \circ m'$, there exists a unique morphism $\psi : C' \to C$ such that $m' = m \circ \psi$, *i.e.*, such that the triangle in Figure 2.7 commutes.

$$\begin{tikzcd} & C' \arrow[dl, dashed, "\psi"'] \arrow[dr, "m'"] & & \\ C \arrow[rr, "m"] & & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Figure 2.7: *Regular monomorphism.*

In Paragraph 2.1 we have given a characterization of monomorphisms and epimorphisms in the categories `Set` and `Poset`. We recall that:

- a monomorphism in `Set` is an injection between sets, a monomorphism in `Poset` is an order-preserving injection between posets;
- an epimorphism in `Set` is a surjection between sets, an epimorphism in `Poset` is an order-preserving surjection between posets.

It is clear from the uniqueness requirement in Definition 2.5 that regular monomorphisms must be monomorphisms. Moreover, in the category `Set` the regular monomorphisms are precisely the injective functions, *i.e.*, the class of regular monomorphisms coincides with the class of monomorphisms. In `Poset` they do not. The following proposition gives a characterization of regular monomorphisms in the category `Poset`.

Proposition 2.3. *In* `Poset`*, regular monomorphisms are precisely order-embeddings.*

Proof. ($\Rightarrow$) Let $(P, \leqslant)$ and $(Q, \leqslant_Q)$ be posets, and let $m : P \to Q$ be a regular monomorphism. By Definition 2.5, there exist $f, g : Q \to R$ such that $f \circ m = g \circ m$. Moreover, since m is mono, by Proposition 2.1, it is an order-preserving injection. Suppose that m is not an order-embedding. Thus, there exist $p_1, p_2 \in P$ such that $m(p_1) \leqslant_Q m(p_2)$, but p_1 and p_2 are incomparable. Consider the poset P' having the same underlying set as P, endowed with the order $\leqslant'$ defined as the transitive closure of the relation obtained by adding to $\leqslant$ the pair (p_1, p_2). This transitive closure does not impair the antisymmetric property. In fact, if we take $a \neq b \in P$, with $a \leqslant b$, the pair $b \leqslant' a$ appears in $\leqslant'$ if and only if $b \leqslant p_1$ and $p_2 \leqslant a$, but in this case we would have $p_2 \leqslant a \leqslant b \leqslant p_1$, against our hypothesis that $p_1 \parallel p_2$.

Consider now the function $\hat{m} : P \to Q$, defined by $\hat{m}(p) = m(p)$, for all $p \in P$. Take $x, y \in P$ such that $x \nleqslant y$ and $x \leqslant' y$. Then, the pair (x, y) of the relation $\leqslant'$ is obtained by the transitive closure described above, and $x \leqslant p_1 \leqslant' p_2 \leqslant y$. Since m is order-preserving, and by hypothesis $m(p_1) \leqslant_Q m(p_2)$, we obtain $\hat{m}(x) \leqslant_Q \hat{m}(p_1) \leqslant_Q \hat{m}(p_2) \leqslant_Q \hat{m}(y)$, and thus $\hat{m}$ can be regarded as an order-preserving map $m' : P' \to Q$. Observing that m' is the same as m on the underlying sets, we immediately obtain $f \circ m' = g \circ m'$.

By the definition of regular monomorphism we can find a unique morphism $\psi : P' \to P$ such that the diagram in Figure 2.8 commutes. Suppose $m(p_1) = m'(p_1) = q_1$ and $m(p_2) =$

$$\begin{tikzcd} & P' \arrow[dl, dashed, "\psi"'] \arrow[dr, "m'"] & & \\ P \arrow[rr, "m"] & & Q \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & R \end{tikzcd}$$

Figure 2.8: *Proof of Proposition 2.3.*

$m'(p_2) = q_2$. We then would have $\psi(p_1) \in m^{-1}(q_1)$ and $\psi(q_2) \in m^{-1}(q_1)$, and so, since m is injective, $\psi(p_1) = p_1$ and $\psi(p_2) = p_2$, but such a ψ would not be order-preserving. Since f and g satisfy $f \circ m = g \circ m$ but are otherwise arbitrary, this would contradict the fact that m is a regular monomorphism. Therefore, m has to be an order-embedding.

($\Leftarrow$) Let $(P, \leqslant)$ and $(Q, \leqslant_Q)$ be posets, and let $m : P \to Q$ be an order-embedding. Consider the poset R, having underlying set $\{0, 1\}$, endowed with the order $\leqslant_R$ such that $0 \parallel 1$. Let $f, g : Q \to R$ be order-preserving maps such that $f(x) = 1$ for all $x \in Q$ and $g(x) = 1$ if and only if $x \in m(P)$. Clearly, $f \circ m = g \circ m$.

Consider now a poset $(P', \leqslant')$ and an order-preserving map $m' : P' \to Q$ such that $f \circ m' = g \circ m'$. It is easy to see that $m'(P') \subseteq m(P)$. Since m is injective, we can construct a function $\hat{\psi} : P' \to P$ by setting $\psi(p) = m^{-1}(m'(p))$, for all $p \in P'$. Such a function can be regarded as an order-preserving map $\psi : P' \to P$, because if $x \leqslant' y$, for $x, y \in P'$, then $m'(x) \leqslant_Q m'(y)$, and, since m is an order-embedding, $m^{-1}(m'(x)) \leqslant m^{-1}(m'(y))$.

Suppose that there exists a morphism $\psi' : P \to P'$, distinct from ψ, such that $m \circ \psi' = m'$. Let $\overline{p}$ be an element of P such that $\psi'(\overline{p}) \neq \psi(\overline{p}) = m^{-1}(m'(\overline{p}))$. Then we would have $m'(\overline{p}) = m(\psi'(\overline{p})) \neq m(\psi(\overline{p})) = m'(\overline{p})$, a contradiction. Summing up, for an arbitrary morphism $m' : P' \to Q$ such that $f \circ m' = g \circ m'$, there exists a unique order-preserving map $\psi : P' \to P$ such that $m \circ \psi = m'$, *i.e.*, m is a regular monomorphism. □

2.3 Regular epimorphisms

The dual notion to regular monomorphism is regular epimorphism.

Definition 2.6. A morphism $e : B \to C$ is called a *regular epimorphism* if and only if there exists a pair $f, g : A \to B$ of morphisms such that:

(1) $e \circ f = e \circ g$,

(2) for any morphism $e' : B \to C'$ with $e' \circ f = e' \circ g$, there exists a unique morphism $\psi : C \to C'$ such that $e' = \psi \circ e$, *i.e.*, such that the triangle in Figure 2.9 commutes.

Clearly, regular epimorphisms are epimorphisms. In particular, in the category `Set` , the class of regular epimorphisms coincides with the class of epimorphisms, *i.e.*, regular epimorphisms are precisely surjective functions. In `Poset` the two classes do not coincide.

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[rr, "e"] \arrow[dr, "e'"'] & & C \arrow[dl, dashed, "\psi"] \\
& & C' &
\end{tikzcd}
$$

Figure 2.9: *Regular epimorphism.*

In order to give a characterization of regular epimorphisms in the category `Poset`, we need some more definitions. First, we recall that a partition of a set A is a set π of nonempty pairwise disjoint subsets of A whose union is A. The members of π are called blocks of π. More details on partitions are given in the following chapters.

Notation. We point out the use of different symbols for representing different types of relations. The symbol $\leqslant$ denotes the partial order relation between elements of a poset. A second symbol, $\lhd$, represents the associated covering relation. Finally, the symbol $\lesssim$ denotes *quasiorder relations*, sometimes called *preorders*, *i.e* reflexive and transitive relations.

Definition 2.7 (Blockwise quasiorder)**.** Let $(P,\leqslant)$ be a poset and let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition of the set P. For $x, y \in P$, *x is blockwise under y with respect to π*, written

$$x \lesssim_\pi y,$$

if and only if there exists a sequence

$$x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y \in P$$

satisfying the following conditions:

(1) for all $i \in \{0, \ldots, n\}$, there exists j such that $x_i, y_i \in B_j$,

(2) for all $i \in \{0, \ldots, n-1\}$, $y_i \leqslant x_{i+1}$.

Observe that the relation $\lesssim_\pi$ in Definition 2.7 indeed is a quasiorder. In fact, if $x \leqslant y$ and $y \leqslant z$ for $x, y, z \in P$, then there exist two sequences $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y$ and $y = y_{n+1}, z_{n+1}, y_{n+2}, z_{n+2}, \ldots, y_{n+m}, z_{n+m} = z$ satisfying (1) and (2), and a sequence $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y_{n+1}, z_{n+1}, y_{n+2}, z_{n+2}, \ldots, y_{n+m}, z_{n+m} = z$ satisfying (1) and (2), too. Thus, $x \lesssim_\pi z$ and the relation $\lesssim_\pi$ is transitive. The reflexivity of $\lesssim_\pi$ results from the reflexivity of $\leqslant$.

Example 2.2. Consider the partially ordered set $(P,\leqslant)$ in Figure 2.10(1) and the partition $\pi_1 = \{B_1, B_2\}$ of P depicted in Figure 2.10(2). The incomparable elements d and a of P are such that $d \lesssim_{\pi_1} a$. In fact, we can build a chain d, b, c, a satisfying (1) and (2) in Definition 2.7. On the other hand, a is not blockwise under d with respect to π_1. In fact, there is no chain of elements of P satisfying (1) and (2) in Definition 2.7. Consider now the partition $\pi_2 = \{C_1, C_2\}$ of P shown in Figure 2.10(3). With respect to π_2, a is blockwise under b and also b is blockwise under a, but $a \neq b$. This shows that $\lesssim_{\pi_2}$ is not a partial order.

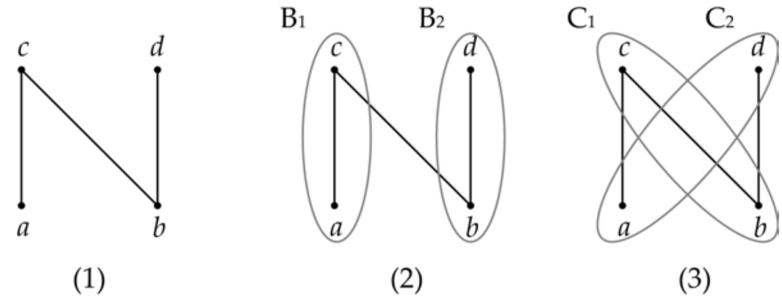


Figure 2.10: *Example 2.2.*

The definition of blockwise quasiorder allows us to isolate a special kind of order-preserving map.

Definition 2.8 (Fibre-coherent map)**.** Consider two partially ordered sets $(P, \leqslant_P)$ and $(Q, \leqslant)$. Let $f : P \to Q$ be a function, and let $\pi_f = \{f^{-1}(q) \mid q \in f(P)\}$ be the set of fibres[2] of f. We say f is a *fibre-coherent map* whenever for any $p_1, p_2 \in P$, $f(p_1) \leqslant f(p_2)$ if and only if $p_1 \lesssim_{\pi_f} p_2$.

A fibre-coherent map is order-preserving. Indeed, if $p_1 \leqslant p_2$ then, by Definition 2.7, $p_1 \lesssim_{\pi_f} p_2$.

Example 2.3. Consider the morphisms f and g shown in Figure 2.11. The map f is not fibre-coherent because $f(d) \leqslant f(c)$, but d is not blockwise under c with respect to the set of fibres π_f. Consider now the set of fibres π_g. Then $d \lesssim_{\pi_g} c$ and the map g is fibre-coherent.

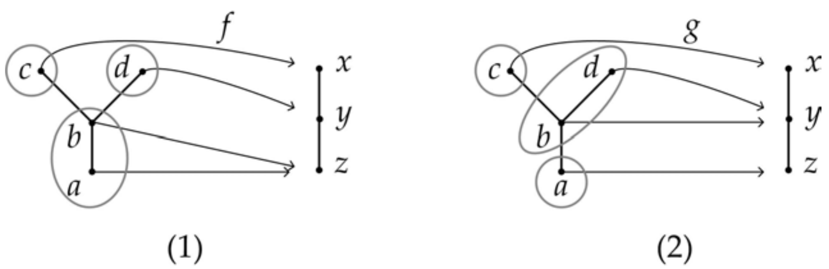


Figure 2.11: *Example 2.3.*

The following proposition gives the promised characterization of regular epimorphisms in the category `Poset`.

[2]Note that π_f is a partition of P.

Proposition 2.4. *In* `Poset`*, regular epimorphisms are precisely fibre-coherent surjections.*

Proof. ($\Rightarrow$) Let $(P, \leqslant_P)$ and $(Q, \leqslant)$ be posets, let $e : P \to Q$ be a regular epimorphism and let $\pi_e = \{e^{-1}(q) \mid q \in Q\}$. Since e is epi, it is an order-preserving surjection by Proposition 2.1. Moreover, by Definition 2.6, there exists a pair $f, g : R \to P$ of morphisms such that $e \circ f = e \circ g$. Suppose that e is not fibre-coherent. If $x \lesssim_{\pi_e} y$ for some $x, y \in P$, then there exists a sequence $x = x_0, y_0, x_1, y_1, \ldots, x_n, y_n = y \in P$ satisfying conditions (1) and (2) in Definition 2.7. For such a sequence, since e is order-preserving, we have $e(x) = e(x_0) = e(y_0) \leqslant e(x_1) = e(y_1) \leqslant \cdots \leqslant e(x_n) = e(y_n) = e(y)$. Thus, there must exist $p_1, p_2 \in P$, with $e(p_1) = q_1$ and $e(p_2) = q_2$, such that $q_1 \leqslant q_2$ but $p_1 \not\lesssim_{\pi_e} p_2$. Note that p_1 and p_2 must be incomparable, and that $q_1 \neq q_2$.

Case (i). Suppose $q_1 \lhd q_2$, where $\lhd$ is the covering relation induced by $\leqslant$. Consider the poset Q' having Q as underlying set, endowed with the relation $\leqslant'$ obtained by removing from $\leqslant$ the pair (q_1, q_2). In other words, the only difference between $\leqslant'$ and $\leqslant$ is that $q_1 \leqslant q_2$, but $q_1 \nleqslant' q_2$. Since $q_1 \lhd q_2$, removing (q_1, q_2) from $\leqslant$ does not impair transitivity and $\leqslant'$ indeed is a partial order.

Now, consider the function $e' : P \to Q'$ that coincides with e on the underlying sets. We want to show that e' is order-preserving. For this, let $x, y \in P$. It suffices to consider two cases only: $e(x) = q_1$ and $e(y) = q_2$, and viceversa. In any other case, e' preserves order just because e does. Suppose, without loss of generality, $x \in e^{-1}(q_1)$ and $y \in e^{-1}(q_2)$. Then, $x \nleqslant_P y$, for else the chain p_1, x, y, p_2 would satisfy conditions (1) and (2) in Definition 2.7, contradicting $p_1 \not\lesssim_{\pi_e} p_2$. Moreover, $y \nleqslant_P x$, because e is order preserving. Thus, for each $x \in e^{-1}(q_1)$ and $y \in e^{-1}(q_2)$, x and y are incomparable. Summing up, e' is order preserving. Since e' coincides with e on the underlying sets, we obtain $e' \circ f = e' \circ g$.

By Definition 2.6 we can find a unique morphism $\psi : Q \to Q'$ such that the diagram in Figure 2.12 commutes. Take $x, y \in P$ such that $e(x) = e'(x) = q_1$ and $e(y) = e'(y) = q_2$. Thus,

$$\begin{tikzcd}
R \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & P \arrow[rr, "e"] \arrow[dr, "e'"'] & & Q \arrow[dl, dashed, "\psi"] \\
& & Q' &
\end{tikzcd}$$

Figure 2.12: *Proof of Proposition 2.4.*

we should have $\psi(q_1) = q_1$ and $\psi(q_2) = q_2$ but, by hypothesis, $q_1 \leqslant q_2$ and $q_1 \nleqslant' q_2$, and such a ψ would not be order-preserving. Since f and g satisfy $e \circ f = e \circ g$ but are otherwise arbitrary, this would contradict the fact that e is a regular epimorphism. Therefore, e has to be fibre-coherent.

Case (ii). Suppose $q_1 \ntriangleleft q_2$. Then there exists a sequence $k_1, k_2, \ldots, k_u \in Q$ such that $q_1 = k_1 \lhd k_2 \lhd \cdots \lhd k_u = q_2$. Let $x_1, x_2, \ldots, x_u \in Q$ be such that $x_i \in e^{-1}(k_i)$, for each $i \in \{1, n\}$, and suppose $x_1 \lesssim_{\pi_e} x_2 \lesssim_{\pi_e} \cdots \lesssim_{\pi_e} x_u$. Since $p_1 \in e^{-1}(k_1)$ and $p_2 \in e^{-1}(k_u)$ imply $p_1 \lesssim_{\pi_e} x_1$ and $x_u \lesssim_{\pi_e} p_2$, then, by transitivity, $p_1 \lesssim_{\pi_e} p_2$, contradicting our hypothesis. Thus, there exists an index j such that $k_j \lhd k_{j+1}$, but $x_j \not\lesssim_{\pi_e} x_{j+1}$. The proof follows now the same steps of *Case* (i), with x_j and x_{j+1} playing the role of p_1 and p_2, respectively.

($\Leftarrow$) Let $(P, \leqslant_P)$ and $(Q, \leqslant)$ be posets, and let $e : P \to Q$ be a fibre-coherent surjection. Consider the poset $R \subseteq P \times P$, having underlying set $\{(r_1, r_2) \in P \times P \mid e(r_1) = e(r_2)\}$, endowed with the order $\leqslant_R$ defined by $(r_1, r_2) \leqslant_R (s_1, s_2)$ if and only if $r_1 \leqslant_P s_1$ and $r_2 \leqslant_P s_2$. Let $f, g : R \to P$ be the projection functions of R, *i.e.*, f and g are the order-preserving maps such that, for each $r = (r_1, r_2) \in R$, $f(r) = r_1$, $g(r) = r_2$. Clearly, $e \circ f = e \circ g$.

We need to show that e is a regular epimorphism. For this, consider a poset $(Q', \leqslant')$ and an order-preserving map $e' : P \to Q'$ such that $e' \circ f = e' \circ g$. Note that, for each $q \in Q$, if $x, y \in e^{-1}(q)$, there exists $r \in R$ such that $f(r) = x$ and $g(r) = y$. Thus, from $e' \circ f = e' \circ g$ follows that $e'(x) = e'(y)$. Since e is a surjection, we can construct a map $\psi : Q \to Q'$ by setting $\psi(q) = e'(x)$ for some $x \in e^{-1}(q)$, where $q \in Q$.

Let now $q_1, q_2 \in Q$ with $q_1 \leqslant q_2$, and let $x_1, x_2 \in P$ be such that $e(x_1) = q_1$, $e(x_2) = q_2$. By Definition 2.8, we have $x_1 \lesssim_{\pi_e} x_2$. Thus, by Definition 2.7 there exists a sequence $y_0, z_0, y_1, z_1, \ldots, y_n, z_n \in P$ with $x_1 = y_0$ and $x_2 = z_n$ such that $e(y_i) = e(z_i)$, for $i = 0, \ldots, n$, and $z_j \leqslant_P y_{j+1}$, for $j = 0, 1, \ldots, n-1$. Moreover, from $e' \circ f = e' \circ g$ it follows that $e'(y_i) = e'(z_i)$, and, since e' is order-preserving, we have $e'(z_j) \leqslant' e'(y_{j+1})$. Thus, $e'(y_0) = e'(z_0) \leqslant' e'(y_1) = e'(z_1) \leqslant' \cdots \leqslant' e'(y_n) = e'(z_n)$. Therefore, we have $\psi(q_1) = e'(x_1) \leqslant \psi(q_2) = e'(x_2)$ and ψ is order-preserving. The morphism ψ is now well defined and, by construction, satisfies $e'(x) = \psi(e(x))$ for all $x \in P$.

Let ψ' be another map from Q to Q', $\psi \neq \psi'$, and let $\overline{q}$ be an element of Q such that $\psi(\overline{q}) \neq \psi'(\overline{q})$. Since e is surjective, there exists $x \in P$ such that $e(x) = \overline{q}$. Then from $\psi'(\overline{q}) \neq \psi(\overline{q})$ and $e'(x) = \psi(\overline{q})$ we have $\psi'(\overline{q}) \neq e'(x)$ and $\psi' \circ e \neq e'$. Hence, $\psi : Q \to Q'$ is the unique function such that $\psi \circ e = e'$. Summing up, for an arbitrary morphism $e' : P \to Q'$ such that $e' \circ f = e' \circ g$, there exists a unique order preserving map $\psi : Q \to Q'$ such that $\psi \circ e = e'$, *i.e.*, e is a regular epimorphism. □

2.4 Factorization systems for `Poset`

To close this categorical chapter we analyze factorization systems for the category `Poset`. We denote by regular epi the class of all regular epimorphisms in a given category, and by regular mono the class of all regular monomorphisms. It is not difficult to see that each of these classes contains all isomorphisms. A first lemma shows another important property of regular epi and regular mono.

Lemma 2.1. *In the category* `Poset`*,* regular mono *and* regular epi *are closed under composition.*

Proof. Trivially, composition of order-preserving maps is order-preserving, composition of surjections is surjective, and composition of injections is injective.

Let $f : A \to B$ and $g : B \to C$ be regular monomorphisms. Then, by Proposition 2.3, f and g are order-embeddings and for each pair $x, y \in A$, $x \leqslant y$ if and only if $f(x) \leqslant f(y)$ in B if and only if $g(f(x)) \leqslant g(f(y))$ in C, so $g \circ f$ is regular mono.

Let now $f : A \to B$ and $g : B \to C$ be regular epimorphisms. By Proposition 2.4, f and g are fibre-coherent surjections. Note that $g \circ f$ is an order-preserving surjection.

Claim (1). For each pair $a_1, a_2 \in A$, $a_1 \lesssim_{\pi_f} a_2$ implies $a_1 \lesssim_{\pi_{g \circ f}} a_2$.
Consider a sequence $x_0, y_0, x_1, y_1, \dots, x_n, y_n$, with $x_0 = a_1$ and $y_n = a_2$, satisfying conditions (1) and (2) in Definition 2.7, with respect to π_f. Since, for all $x, y \in A$, $f(x) = f(y)$ implies $g(f(x)) = g(f(y))$, the same sequence satisfies the same conditions with respect to $\pi_{g \circ f}$.

Claim (2). For each pair $a_1, a_2 \in A$, $f(a_1) \lesssim_{\pi_g} f(a_2)$ implies $a_1 \lesssim_{\pi_{g \circ f}} a_2$.
Consider a sequence $x_0, y_0, x_1, y_1, \dots, x_n, y_n$, with $x_0 = f(a_1)$ and $y_n = f(a_2)$, satisfying conditions (1) and (2) in Definition 2.7, with respect to π_g. Construct a sequence of elements of A, $x'_0, y'_0, x'_1, y'_1, \dots, x'_n, y'_n$, such that $x'_i \in f^{-1}(x_i)$, $y'_i \in f^{-1}(y_i)$, for all $i = 0, \dots, n$, and $x'_0 = a_1$, $y'_n = a_2$. Since, for all i, $g(x_i) = g(y_i)$, we have $g(f(x'_i)) = g(f(y'_i))$. Moreover, as $y_j \leqslant x_{j+1}$ for all $j = 0, \dots, n-1$ and f is fibre-coherent, we have $y'_j \lesssim_{\pi_f} x'_{j+1}$. By Claim (1), we obtain $y'_j \lesssim_{\pi_{g \circ f}} x'_{j+1}$. It is now easy to expand our sequence of elements of A to a sequence satisfying conditions (1) and (2) in Definition 2.7, with respect to $\pi_{g \circ f}$, and we obtain $a_1 \lesssim_{\pi_{g \circ f}} a_2$.

Claim (3). For each pair $a_1, a_2 \in A$, $g(f(a_1)) \leqslant g(f(a_2))$ implies $a_1 \lesssim_{\pi_{g \circ f}} a_2$.
Since $g(f(a_1)) \leqslant g(f(a_2))$ and g is fibre-coherent, we have $f(a_1) \lesssim_{\pi_g} f(a_2)$ and, by Claim (2), $a_1 \lesssim_{\pi_{g \circ f}} a_2$.

Claim (4). For each pair $a_1, a_2 \in A$, $a_1 \lesssim_{\pi_{g \circ f}} a_2$ implies $g(f(a_1)) \leqslant g(f(a_2))$.
Consider a sequence $x_0, y_0, x_1, y_1, \dots, x_n, y_n$, with $x_0 = a_1$ and $y_n = a_2$, satisfying conditions (1) and (2) in Definition 2.7, with respect to $\pi_{g \circ f}$. Since $g \circ f$ is order-preserving, we have $g(f(x_0)) = g(f(y_0)) \leqslant g(f(x_1)) = g(f(y_1)) \leqslant \cdots \leqslant g(f(x_n)) = g(f(y_n))$, that is $g(f(a_1)) \leqslant g(f(a_2))$.

By Claims (3) and (4), $g \circ f$ is fibre-coherent. □

At this point, what we have is four classes of morphisms (including epi and mono), closed under composition, and each containing all isomorphisms. These are our candidates for a factorization system in Poset. After another preparatory lemma, we will be ready to show that there are two different factorization systems for our category, each dual to the other.

Lemma 2.2. *If each morphism of a category $\mathcal{K}$ has* (regular epi,mono) *factorization, then this factorization is essentially unique. Dually, if each morphism of a category $\mathcal{K}$ has* (epi,regular mono) *factorization, then this factorization is essentially unique.*

Proof. We refer to Figure 2.13. Suppose that, for a morphism $f : A \to C$, $m \circ e = f = m' \circ e'$, with e, e' regular epimorphisms and m, m' monomorphisms. Then, there exists a, b with $e \circ a = e \circ b$ satisfying condition (2) in Definition 2.6. As $m' \circ e' \circ a = m \circ e \circ a = m' \circ e' \circ b = m \circ e \circ b$, and m' is mono, $e' \circ a = e' \circ b$. Therefore, there exists a unique g such that $g \circ e = e'$. Since e is epi, $m' \circ g = m$. Symmetrically, there exists $h : B' \to B$ with $h \circ e' = e$ and $m \circ h = m'$.

$$
\begin{tikzcd}
 & & B \arrow[dd, dashed, shift right, "g"'] \arrow[dr, "m"] & \\
{} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & A \arrow[ur, "e"] \arrow[dr, "e'"'] & & C \\
 & & B' \arrow[uu, dashed, shift right, "h"'] \arrow[ur, "m'"'] &
\end{tikzcd}
$$

Figure 2.13: *Proof of Lemma 2.1.*

Since e is epi (or since m is mono), we infer $h \circ g = id_B$. Symmetrically, $g \circ h = id_{B'}$. The proof of the second statement of the lemma is analogous. □

Proposition 2.5. (epi,regular mono) *is a factorization system for the category* `Poset`. *Dually,* (regular epi,mono) *is a factorization system for the category* `Poset`.

Proof. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be posets, and let $f : P \to Q$ be a morphism. Let $\hat{f} : P \to Q$ be the function such that $\hat{f}(p) = f(p)$ for all $p \in P$, and consider the poset $(\hat{f}(P), \leqslant)$, where $\leqslant$ is the restriction of $\leqslant_Q$ to $\hat{f}(P)$, *i.e.*, for all $x, y \in \hat{f}(P)$, $x \leqslant y$ if and only if $x \leqslant_Q y$. Then, consider the functions $\hat{e} : P \to \hat{f}(P)$, defined by $\hat{e}(p) = f(p)$, for all $p \in P$, and $\hat{m} : \hat{f}(P) \to Q$, defined by $\hat{m}(q) = q$, for all $q \in \hat{f}(P)$. By construction, $\hat{e}$ can be regarded as an order-preserving surjection $e : P \to \hat{f}(P)$ and $\hat{m}$ can be regarded as an order-embedding $m : \hat{f}(P) \to Q$. Moreover, $f = m \circ e$. Therefore, in `Poset`, every morphism has an (epi,regular mono) factorization. Lemma 2.2 guarantees that this factorization is essentially unique, and, by Lemma 2.1, (epi,regular mono) is a factorization system for `Poset`.

Let $P, Q, f, \hat{f}$ be as in the above, and consider a poset $(\hat{f}(P), \leqslant)$, where $\leqslant$ is the partial order on $\hat{f}(P)$ such that for all $x, y \in \hat{f}(P)$, $x \leqslant y$ if and only if $x \leqslant_Q y$ and for each $p_1 \in f^{-1}(x), p_2 \in f^{-1}(y)$, $p_1 \lesssim_{\pi_f} p_2$. Consider now the functions $\hat{e} : P \to \hat{f}(P)$, defined by $\hat{e}(p) = f(p)$, for all $p \in P$, and $\hat{m} : \hat{f}(P) \to Q$, defined by $\hat{m}(q) = q$, for all $q \in \hat{f}(P)$. By construction, $\hat{e}$ can be regarded as a fibre-coherent surjection $e : P \to \hat{f}(P)$ and $\hat{m}$ can be regarded as an order-preserving injection $m : \hat{f}(P) \to Q$. Moreover, $f = m \circ e$. Therefore, in `Poset`, every morphism has a (regular epi,mono) factorization. Lemma 2.2 guarantees that this factorization is essentially unique, and, by Lemma 2.1, (regular epi,mono) is a factorization system for `Poset`. □

The following example shows how to factor a given morphism in `Poset`.

Example 2.4. Let $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ be the posets represented in Figure 2.14. Let $g : P \to Q$ be the order-preserving map defined by $g(a) = y$, $g(b) = g(d) = x$, $g(c) = z$. Let $\hat{g} : P \to Q$ be the function such that $\hat{g}(p) = g(p)$ for all $p \in P$.

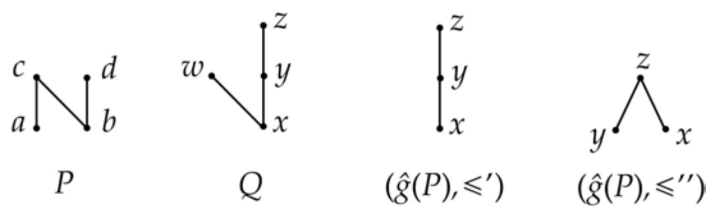


Figure 2.14: *Example 2.4.*

Consider the poset $(\hat{g}(P), \leqslant')$, where $\leqslant'$ is the restriction of $\leqslant_Q$ to $\hat{g}(P)$, and the morphisms $e : P \to \hat{g}(P)$ such that $e(a) = y$, $e(b) = e(d) = x$, $e(c) = z$, and $m : \hat{g}(P) \to Q$ such that $m(x) = x$, $m(y) = y$, $m(z) = z$. Then, e is epi, m is regular mono, and g factors as $g = m \circ e$.

Consider now the poset $(\hat{g}(P), \leqslant'')$, where $\leqslant''$ is the partial order on $\hat{g}(P)$ such that for all $x, y \in \hat{g}(P)$, $x \leqslant'' y$ if and only if $x \leqslant_Q y$ and for each $p_1 \in g^{-1}(x), p_2 \in g^{-1}(y)$, $p_1 \lesssim_{\pi_g} p_2$. Let $e' : P \to \hat{g}(P)$ such that $e'(a) = y$, $e'(b) = e'(d) = x$, $e'(c) = z$, and $m' : \hat{g}(P) \to Q$ such that $m'(x) = x$, $m'(y) = y$, $m'(z) = z$. Then, e' is regular epi, m' is mono, and g factors as $g = m' \circ e'$.

2.5 Bibliographic notes

The categorical notions used in this chapter can be found, in different variants, in almost every book concerned with categories. We cite, for example, [AHS04], [AM75], [HS73], [Man76], [GZ02].

For our definitions of isomorphism, epimorphism, monomorphism, regular epimorphism and regular monomorphism, we refer, in particular, to [AHS04, Ch. II.7 and Def. 3.8]. The formal definition of category is taken from [AM75, Pag. 29], and the proof of Lemma 2.2 is taken from [Man76, Prop. 1.49].

Chapter IV of [AHS04] gives a thorough analysis of factorization systems in general categories.

Chapter 3

Partitioning a poset

...as bit by bit it starts the need to just let go my party piece.

Robert Smith

3.1 Partitions: from sets to posets

In the previous chapter, we have introduced set-theoretic partitions. A formal definition of partition of a set is given in the following.

Definition 3.1. A *partition* of a set A (also called *set partition*) is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of subsets of A such that, for each $i, j \in \{1, \ldots, k\}$,

a. $B_i \neq \emptyset$,

b. $B_i \cap B_j = \emptyset$ for $i \neq j$,

c. $B_1 \cup B_2 \cup \cdots \cup B_k = A$.

We call B_i a *block* of π, and we say that π *has k blocks*, *i.e.*, $|\pi| = k$.

There are many other ways to define partitions. For instance, one can define partitions by means of equivalence relations. This will be useful in the next chapter. A definition of partitions can also be given in terms of functions between sets, as follows.

Definition 3.2. A *partition* of a set A is the set π_f of fibres of a surjection $f : A \to C$, for some set C.

It is an exercise to check that partitions in the sense of Definition 3.2 coincide with partitions as introduced in Definition 3.1.

Example 3.1. Let $A = \{a, b, c, d, e\}$ be a set, and consider a surjection $f : A \to C$, with $C = \{x, y, z\}$ and f defined as $f(a) = x$, $f(b) = f(c) = y$, $f(d) = f(e) = z$. The set of fibres $\pi_f = \{\{a\}, \{b, c\}, \{d, e\}\}$ is a partition of A.

To arrive at a definition of partition of a partially ordered set, we focus on partitions of a set as collection of fibres. The question arises, which kind of "surjections" should we consider in adapting Definition 3.2 to posets? The answer to this question is given by the previous chapter. We should consider two different kinds of surjections between posets,

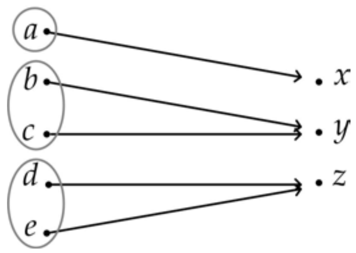


Figure 3.1: *Example 3.1.*

namely, order-preserving surjections, *i.e.*, epimorphisms, and fibre-coherent surjections, *i.e.*, regular epimorphisms. From these two types of surjections we will obtain two different notions of partitions.

Notation. From now on, we make use of the symbol $\preccurlyeq$ to denote a partial order between blocks of a partition. As in the previous chapter, the symbol $\leqslant$ denotes the partial order relation between elements of a poset, and the symbol $\lesssim$ denotes quasiorder relations. In particular, we use the symbol $\lesssim_\pi$ to denote a blockwise quasiorder with respect to a partition π, as defined in Definition 2.7.

Definition 3.3. A *monotone partition* of a poset P is a poset $(\pi_f, \preccurlyeq)$, where π_f is the set of fibres of an order-preserving surjection $f : P \to Q$, for some poset Q, and $\preccurlyeq$ is the partial order on π_f defined by

$$f^{-1}(q_1) \preccurlyeq f^{-1}(q_2) \text{ if and only if } q_1 \leqslant q_2\,, \tag{3.1}$$

for each $q_1, q_2 \in Q$.

Example 3.2. Consider the morphism $f : P \to Q$ between posets depicted in Figure 3.2. Since f is an order-preserving surjection, the poset $(\pi_f, \preccurlyeq)$ as in Definition 3.3 is a monotone partition of P.

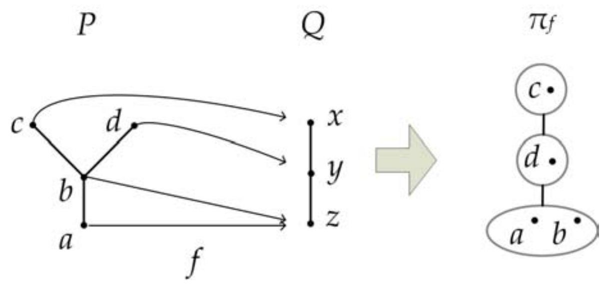


Figure 3.2: *Example 3.2.*

Definition 3.4. A *regular partition* of a poset P is a poset $(\pi_f, \leqslant)$, where π_f is the set of fibres of a fibre-coherent surjection $f : P \to Q$, for some poset Q, and $\leqslant$ is the partial order on π_f defined by

$$f^{-1}(q_1) \leqslant f^{-1}(q_2) \text{ if and only if } q_1 \leqslant q_2\,, \tag{3.2}$$

for each $q_1, q_2 \in Q$.

Remark 3.1. Each regular partition is a monotone partition, just because each fibre-coherent surjection is an order-preserving surjection (see Section 2.3).

Example 3.3. Consider the morphism $g : P \to Q$ between posets depicted in Figure 3.3. Since the morphism g is a fibre-coherent surjection, the poset $(\pi_g, \leqslant)$, where π_g is the set of fibres of g and $\leqslant$ is the partial order induced by g according to Definition 3.4, is a regular partition of P. Note that the poset $(\pi_f, \leqslant)$ in Figure 3.2 is not a regular partition of P, because f is not a fibre-coherent map (see Example 2.3).

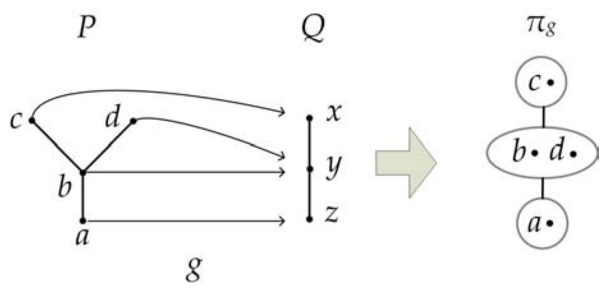


Figure 3.3: *Example 3.3.*

Returning for a moment to sets, we note that while Definition 3.1 and Definition 3.2 are indeed equivalent, the former is definitely more common. Indeed, since Definition 3.1 does not mention morphisms, it may be considered more readable. Besides, it allows us to visualize a partition just by looking at the set we want to partition.

What we need now, and what we will give in the coming sections, are intrinsic characterizations of the two kinds of partitions of a poset that we have introduced in Definitions 3.3 and 3.4, namely, monotone and regular partitions.

3.2 Monotone partitions

The following theorem gives a characterization of monotone partitions of partially ordered sets.

Theorem 3.1. *If P is a poset, $(\pi = \{B_1, B_2, \ldots, B_k\}, \leqslant)$ is a monotone partition of P if and only if π is a partition of the underlying set of P, and $\leqslant$ is a partial order on π such that for* *each pair B_i, B_j of blocks of π, and for all $x \in B_i$, $y \in B_j$,*

$$x \preccurlyeq y \text{ implies } B_i \preccurlyeq B_j\,. \tag{3.3}$$

Proof. ($\Leftarrow$) Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition of the underlying set of a poset P, and let $\preccurlyeq$ be a partial order on π satisfying Condition (3.3). Consider the function $\hat{f} : P \to \pi$ which sends each element of P to its block in π. Clearly, $\hat{f}$ is a surjection, because a partition does not have empty blocks. Moreover, by Condition (3.3), the function $\hat{f}$ can be regarded as an order-preserving surjection $f : P \to (\pi, \preccurlyeq)$, having π as its set of fibres. Since $f^{-1}(B_i) = B_i$ for each $i \in \{1, \ldots, k\}$, the partial order $\preccurlyeq$ satisfies Condition (3.1) in Definition 3.3, and $(\pi, \preccurlyeq)$ is a monotone partition of P.

($\Rightarrow$) Let $f : P \to Q$ be an order-preserving surjection, and let $(\pi_f = \{B_1, B_2, \ldots, B_k\}, \preccurlyeq)$ be a monotone partition of the poset P. Since the function f is surjective, then π_f is a partition of the underlying set of P. Consider now $x \in B_i$ and $y \in B_j$, with $i, j \in \{1, \ldots, k\}$, such that $x \preccurlyeq y$ in P. Since f is order-preserving, $f(x) \preccurlyeq f(y)$ holds and, by Definition 3.3, $B_i = f^{-1}(f(x)) \preccurlyeq B_j = f^{-1}(f(y))$. □

Example 3.4. Figure 3.4 shows three different monotone partitions of the poset P. It is easy to check that Condition (3.3) holds for each of the partial orders $\preccurlyeq_1$, $\preccurlyeq_2$, $\preccurlyeq_3$.

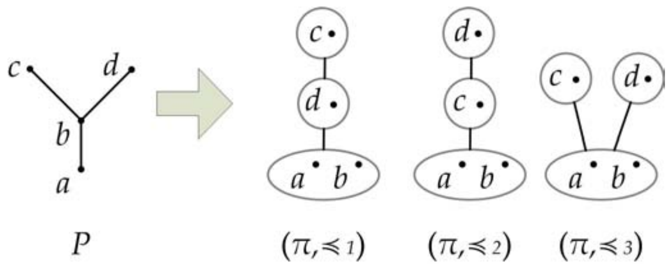


Figure 3.4: *Example 3.4.*

3.3 Regular partitions

The following theorem gives a characterization of regular partitions of partially ordered sets.

Theorem 3.2. *If P is a poset, $(\pi = \{B_1, B_2, \ldots, B_k\}, \preccurlyeq)$ is a regular partition of P if and only if π is a partition of the underlying set of P, and $\preccurlyeq$ is a partial order on π such that for each pair B_i, B_j of blocks of π, and for all $x \in B_i$, $y \in B_j$,*

$$x \lesssim_\pi y \text{ if and only if } B_i \preccurlyeq B_j\,, \tag{3.4}$$

where $\lesssim_\pi$ is the blockwise quasiorder induced by π.

Proof. ($\Leftarrow$) Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a partition of the underlying set of a poset P and let $\preccurlyeq$ be a partial order on π satisfying Condition (3.4). Consider the surjection $\hat{f} : P \to \pi$ which sends each element of P to its block in π. We can immediately note that for the function $\hat{f}$ Condition (3.4) is equivalent to the fibre-coherent condition in Definition 2.8, and $\hat{f}$ can be regarded as a fibre-coherent surjection $f : P \to (\pi, \preccurlyeq)$, having π as its set of fibres. Moreover, since $f^{-1}(B_i) = B_i$ for all i, the partial order $\preccurlyeq$ satisfies Condition (3.2) in Definition 3.4 and $(\pi, \preccurlyeq)$ is a regular partition of P.

($\Rightarrow$) Let $f : P \to Q$ be a fibre-coherent surjection, and let $(\pi_f = \{B_1, B_2, \ldots, B_k\}, \preccurlyeq)$ be a regular partition of the poset P. Since the function f is surjective, then π_f is a partition of the underlying set of P. Consider $x \in B_i$ and $y \in B_j$, with $i, j \in \{1, \ldots, k\}$. By Definition 2.8, $x \lesssim_{\pi_f} y$ if and only if $f(x) \preccurlyeq f(y)$. By Definition 3.4, $f(x) \preccurlyeq f(y)$ if and only if $B_i = f^{-1}(f(x)) \preccurlyeq B_j = f^{-1}(f(y))$. Thus, $x \lesssim_{\pi_f} y$ if and only if $B_i \preccurlyeq B_j$. □

Example 3.5. Figure 3.4 shows a regular partition of the poset P. Condition (3.4) holds. In fact, $d \lesssim_\pi c$, and the other elements satisfy the condition trivially. If we look at the previous example, Figure 3.4, one can easily check that $\preccurlyeq_1$ and $\preccurlyeq_2$ do not satisfy Condition (3.4), while $(\pi, \preccurlyeq_3)$ is regular.

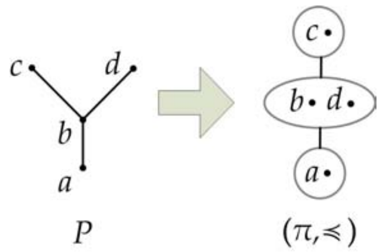


Figure 3.5: *Example 3.5.*

3.4 Extending set partitions to monotone partitions

Theorems 3.1 and 3.2 characterize monotone and regular partitions. They allow us to construct partitions of a poset just by looking at the poset itself, without considering morphisms. Nevertheless, constructing and recognizing such partitions remains a nontrivial task. In this section, we give some corollaries which help simplifying this task. A first corollary introduces a new condition useful to construct partitions of posets starting from partitions of the underlying sets. In other words, we show how to *extend* a set partition π to a partition of a poset P, endowing π with an order $\preccurlyeq$ between blocks that makes $(\pi, \preccurlyeq)$ become a monotone partition of P.

Corollary 3.3. *Let P be a poset, and π a set partition of P. Then, π admits an extension to a monotone partition of the poset P if and only if for all $x, y \in P$, $x \lesssim_\pi y$ and $y \lesssim_\pi x$ imply that there is $B \in \pi$ such that $x, y \in B$.*

Proof. (⇒) Suppose that the set partition $\pi = \{B_1, \dots, B_k\}$ of P can be extended to a monotone partition $(\pi, \preccurlyeq)$. By Theorem 3.1 we can construct an order-preserving surjection $f : P \to \pi$ such that the set π_f of the fibres of f coincides with π. Let $x \lesssim_\pi y$ and $y \lesssim_\pi x$, for some $x, y \in P$, with $x \in B_i$ and $y \in B_j$. Then, there exist two sequences $x = x_0, y_0, x_1, y_1, \dots, x_n, y_n = y$ and $y = z_0, w_0, \dots, z_m, w_m = x$ satisfying Conditions (1) and (2) in Definition 2.7 with respect to π. Since f is order-preserving, we have

$B_i = f(x) = f(x_0) = f(y_0) \preccurlyeq f(x_1) = f(y_1) \preccurlyeq \cdots \preccurlyeq f(x_n) = f(y_n) = f(y) = B_j$, and

$B_j = f(y) = f(z_0) = f(w_0) \preccurlyeq f(z_1) = f(w_1) \preccurlyeq \cdots \preccurlyeq f(z_m) = f(w_m) = f(x) = B_i$.

Thus, we have $B_i = B_j$.

(⇐) Let $\pi = \{B_1, \dots, B_k\}$ be a set partition of P such that for all $x, y \in P$, $x \lesssim_\pi y$ and $y \lesssim_\pi x$ imply $x, y \in B \in \pi$. Define the binary relation $\preccurlyeq$ on π by prescribing that for all B_i, $B_j \in \pi$, and for all $x \in B_i$, $y \in B_j$, $x \lesssim_\pi y$ if and only if $B_i \preccurlyeq B_j$. It is immediate to check that $\preccurlyeq$ is a partial order. By Theorem 3.2, $(\pi, \preccurlyeq)$ is a monotone partition – in fact, a regular one. □

For regular partitions one can say more: a set partition of P admit at most one extension to a regular partition of the poset P. The results for regular partitions are even stronger. First, we have that the same condition as in Corollary 3.3 is the necessary condition for a set partition π to be extended to a partition of a poset. Second, if this extension exists, it is unique.

Corollary 3.4. *Let P be a poset, and π a set partition of P. Then, π admits an extension to a regular partition of the poset P if and only if for all $x, y \in P$, $x \lesssim_\pi y$ and $y \lesssim_\pi x$ imply that there exists $B \in \pi$ such that $x, y \in B$. In case such an extension exists, it is unique.*

Proof. (⇒) Each regular partition is a monotone partition, and the result follows directly from Corollary 3.3.

(⇐) The monotone partition $(\pi, \preccurlyeq)$ of P constructed in the proof of Corollary 3.3 is regular. To see that it is unique, consider an extension of the set partition $\pi = \{B_1, \dots, B_k\}$ of P to a regular partition $(\pi, \preccurlyeq')$ of P. Then, $\preccurlyeq'$ must be such that for each B_i, $B_j \in \pi$, and for all $x \in B_i$, $y \in B_j$, $x \lesssim_\pi y$ if and only if $B_i \preccurlyeq B_j$, for else we would violate the necessary Condition (3.4) in Theorem 3.2. □

For "small" posets P, using these two corollaries it is easy to check whether a set partition π of the underlying set of P can be extended to a poset partition of P.

Example 3.6. Consider the poset P shown in Figure 3.6 with the depicted set partition $\pi = \{\{a, x\}, \{b, z\}, \{c, y\}, \{d, w\}\}$. The elements a, d of P are such that $a \lesssim_\pi d$ (witness the sequence a, a, y, c, w, d) and $d \lesssim_\pi a$ (witness the sequence d, d, z, b, x, a), but a and d are not in the same block. Thus, there is no partition of P having π as its underlying set.

Example 3.7. We show by example that the uniqueness assertion in Corollary 3.4 fails for monotone partitions. Consider the trivially ordered set P shown in Figure 3.7. The underlying set of P has only two set partitions, each extendable to a partition of the poset P. The partition $\{\{a\}, \{b\}\}$ has one extension (Figure 3.7(1)) to a regular partition, and three

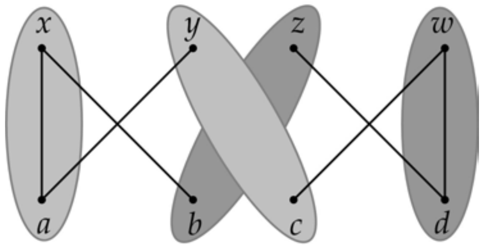


Figure 3.6: *Example 3.6.*

extensions (Figure 3.7(2)) to monotone partitions. The partition $\{\{a,b\}\}$ can be extended to a monotone partition in just one way.

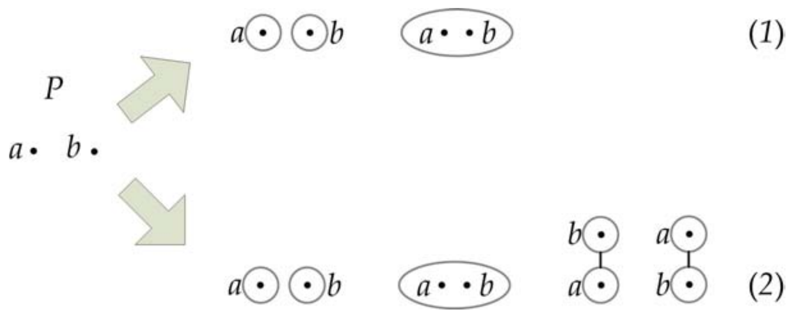


Figure 3.7: *Example 3.7.*

Example 3.8. Figure 3.8 shows the 14 monotone partitions of the poset P (the same used in Examples 2.3, 3.2, 3.3, 3.4, and 3.5). Regular partitions are shaded.

In the next chapter, we give a lattice structure to the collection of monotone (or regular) partitions of a poset P. To achieve this, we shall give a representation of partitions of posets in terms of quasiorders.

3.5 Bibliographic notes

While set partitions have been discussed in many books (*e.g.*, [Com74],[Sta97], [Aig79]) and papers (*e.g.*, [Rot64a]), the literature on partitioning partially ordered sets is rather scant. In particular, we are not aware of the previous appearance of Definitions 3.3 and 3.4 and the related thing. However, a significative item is [Ric98], where the author investigates partially ordered set-theoretic partitions, which turn out to be monotone partitions (in our sense) of trivially ordered sets.

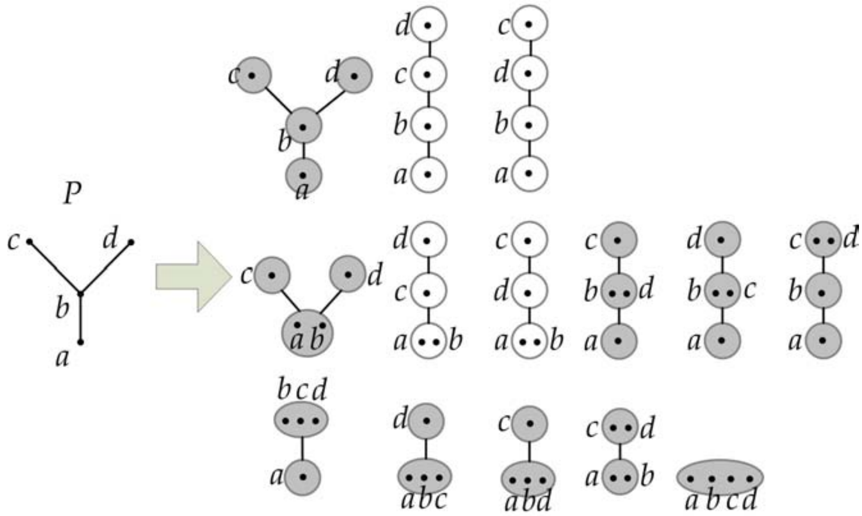


Figure 3.8: *Example 3.8.*

Chapter 4

The lattices of partitions

Order is the shape upon which beauty depends.

Pearl Buck

4.1 Partitions as quasiorders

A well-known elementary result establishes a connection between set partitions and equivalence relations, *i.e.*, reflexive, transitive, and symmetric relations. This connection can be expressed as follows.

Fact 4.1. *Any partition π of a set A determines the equivalence relation $\equiv$ on A such that $a \equiv b$ whenever a and b belong to the same block of π. Conversely, any equivalence relation $\equiv$ on a set A determines the partition π of A whose blocks consist of the equivalence classes of $\equiv$.*

Thanks to this connection, when studying the structure of all partitions of a given set, one can look at the collection of equivalence relations on that set, and viceversa. Let us indicate with Π the collection of all partitions of a set A. Π is partially ordered by

$\pi_1 \leqslant \pi_2$ if and only if each block in π_1 is obtained by subdivision of the blocks in π_2.

If $\pi_1 \leqslant \pi_2$ holds, we say that π_1 is a *refinement* of π_2. We can observe that the poset $(\Pi, \leqslant)$ has both bottom and top elements. The bottom element is the partition of A having all singleton blocks. The top element is the one block partition. Equivalently, we can consider Π to be the collection of all equivalence relations on A. A partial order on this set can be defined as the set-theoretic inclusion between relations. The poset obtained has as bottom element the identity relation, *i.e.*, the relation $\equiv_I$ such that for any $a, b \in A$, $a \equiv_I b$ if and only if $a = b$. The top element is the universal relation, *i.e.*, the relation $\equiv_U$ such that for any $a, b \in A$, $a \equiv_U b$. A fundamental result for the structure of set partitions, or equivalence relations, is the following.

Fact 4.2. *The collection of all partitions or all equivalence relations of a set A forms a complete lattice. We call this lattice the* partition lattice *of A.*

We seek to generalize the relationship between equivalence relations and partitions introduced in Fact 4.1 to an analogous relationship between partitions of partially ordered sets and quasiorder relations, *i.e.*, reflexive and transitive relations. The aim is to obtain the analogous of Fact 4.2 for monotone and regular partitions.

Notation. Recall that, as in the previous chapters, we use the symbol $\lesssim$ to denote quasiorder relations. Whenever $\lesssim$ is used to denote a blockwise quasiorder (see Definition 2.7), it is denoted $\lesssim_\pi$, where π is a partition.

The following result holds for monotone partitions.

Proposition 4.1. *There is a bijection between monotone partitions of a poset* $(P, \preccurlyeq)$ *and quasiorders on* P *extending* $\preccurlyeq$.

Specifically, we can construct this bijection associating with each monotone partition $(\pi, \preccurlyeq)$ *a quasiorder* $\lesssim$ *by setting, for any* $B, C \in \pi$, *and for* $x \in B$, $y \in C$,

$$B \preccurlyeq C \text{ if and only if } x \lesssim y. \tag{4.1}$$

To the underlying partition π *there corresponds the set of equivalence classes of the equivalence relation* $\equiv$ *induced by* $\lesssim$, *that is, the equivalence relation defined as*

$$x \equiv y \text{ if and only if } x \lesssim y \text{ and } y \lesssim x, \tag{4.2}$$

for any $x, y \in P$.

Proof. A quasiorder $\lesssim$ on a set P may allow both $x \lesssim y$ and $y \lesssim x$ to hold for certain pairs $x, y \in P$. Then, we can define a relation $\equiv$ on P such that

$$x \equiv y \text{ if and only if } x \lesssim y \text{ and } y \lesssim x,$$

and immediately notice that $\equiv$ is an equivalence relation. Thus, the quasiorder $\lesssim$ defines a set of equivalence classes $[x]_\equiv = \{y \in P \mid x \equiv y\}$. By Fact 4.1, the set $\pi = \{[x]_\equiv \mid x \in P\}$ is a partition of P. Consider now the relation $\preccurlyeq$ on π defined, for $B, C \in \pi$, by

$$B \preccurlyeq C \text{ if and only if } x \lesssim y, \text{ for each } x \in B,\ y \in C.$$

Observe that $B \preccurlyeq C$ and $C \preccurlyeq B$ hold if and only if $x \lesssim y$ and $y \lesssim x$, that is if and only if $B = C$. One can easily check that $\preccurlyeq$ is a partial order on π. Moreover, by hypothesis, for each $B, C \in \pi$, and for any $x \in B$, $y \in C$, $x \preccurlyeq y$ implies $x \lesssim y$. Thus, $\preccurlyeq$ satisfies Condition (3.3) in Theorem 3.1 and $(\pi, \preccurlyeq)$ is a monotone partition of $(P, \preccurlyeq)$.

Conversely, let $(\pi, \preccurlyeq)$ be a monotone partition of a poset $(P, \preccurlyeq)$. Define a relation $\lesssim$ on P as follows, for $x \in B$, $y \in C$, with $B, C \in \pi$.

$$x \lesssim y \text{ if and only if } B \preccurlyeq C.$$

Observe that, by Condition (3.3) in Theorem 3.1, $x \preccurlyeq y$ implies $B \preccurlyeq C$. We immediately see that $\lesssim$ is a quasiorder on P which extends $\preccurlyeq$. Moreover, according to Fact 4.1, the equivalence relation induced by $\lesssim$ defines precisely the partition π. □

We give an example of the connection between monotone partitions and quasiorders.

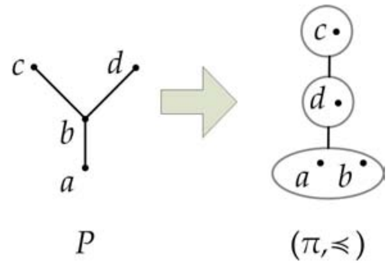


Figure 4.1: *Example 4.1.*

Example 4.1. Figure 4.1 shows a monotone partition π of a poset P.
The corresponding quasiorder defined as in Proposition 4.1 is

$$\lesssim \, = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(a,c),(b,c),(a,d),(b,d),(d,c)\}.$$

Conversely, let us begin with the quasiorder $\lesssim$ above. Then $\lesssim$ induces the equivalence relation

$$\equiv \, = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}$$

on the set P, and thus a partition π of P. The order relation $\preccurlyeq$ on the blocks of π, defined as in Proposition 4.1, is

$$\preccurlyeq \, = \{([a]_\equiv,[a]_\equiv),([c]_\equiv,[c]_\equiv),([d]_\equiv,[d]_\equiv),([a]_\equiv,[c]_\equiv),([a]_\equiv,[d]_\equiv),([d]_\equiv,[c]_\equiv)\}.$$

An analogous connection can be obtained for regular partitions. Let us denote by $\mathrm{tr}(R)$ the transitive closure of a binary relation R.

Proposition 4.2. *Let $(\pi = \{B_1, B_2, \dots, B_k\}, \preccurlyeq)$ be the monotone partition of $(P, \leqslant)$ induced by a quasiorder $\lesssim$ on P as in Proposition 4.1. Consider the binary relation $\rho = \{(x,y) \in P \times P \mid x \lesssim y,\ x \nleqslant y,\ y \not\lesssim x\}$. Then π is a regular partition of $(P, \leqslant)$ if and only if*

$$\lesssim \, = \, \mathrm{tr}(\lesssim \setminus \rho). \tag{4.3}$$

Proof. Let $\lesssim' = \mathrm{tr}(\lesssim \setminus \rho)$.

Suppose that $p \lesssim_\pi q$, for $p, q \in P$, with the intent of showing $p \lesssim' q$. Then, by Definition 2.7, there exists a sequence $x_0, y_0, x_1, y_1, \dots, x_n, y_n \in P$, with $p = x_0$ and $q = y_n$, satisfying the following.

1. For all $i \in \{0, \dots, n\}$, there exists j such that $x_i, y_i \in B_j$.
2. For all $i \in \{0, \dots, n-1\}$, $y_i \leqslant x_{i+1}$.

By the first condition, $x_i \lesssim y_i$ and $y_i \lesssim x_i$. Thus, $(x_i, y_i), (y_i, x_i) \notin \rho$ and $x_i \lesssim' y_i$, $y_i \lesssim' x_i$. By 2., $y_i \leqslant x_{i+1}$, thus $(y_i, x_{i+1}) \notin \rho$. Since $y_i \lesssim x_{i+1}$, we also have $y_i \lesssim' x_{i+1}$. Therefore, we have $x_0 \lesssim' y_0 \lesssim' x_1 \lesssim' y_1 \lesssim' \cdots \lesssim' x_n \lesssim' y_n$ and, by transitivity, $p \lesssim' q$.

Suppose now that, for some $p,q \in P$, we have $p \lesssim' q$, with the intent of showing $p \lesssim_\pi q$. Observe that $p \lesssim q$, because $\lesssim' \subseteq \lesssim$. We want to construct a sequence $p = x_0 \lesssim' y_0 \lesssim' x_1 \lesssim' y_1 \lesssim' \cdots \lesssim' x_n \lesssim' y_n = q$ of elements of P such that, for all $i \in \{0,\dots,n\}$, $x_i \lesssim y_i$ and $y_i \lesssim x_i$, and, for all $j \in \{0,\dots,n-1\}$, $y_j \leqslant x_{j+1}$. We shall analyze three different cases, covering all the possibilities for p and q.

(Case 1). $p \lesssim q$ and $q \lesssim p$. In this case we can set $x_0 = p$ and $y_0 = q$, thus obtaining the desired sequence, with $n = 0$.

(Case 2). $p \leqslant q$, but $q \not\lesssim p$. In this case we can set $x_0 = y_0 = p$ and $x_1 = y_1 = q$, obtaining the desired sequence, with $n = 1$.

(Case 3). $(p,q) \in \rho$, *i.e.* $p \lesssim q$, $p \not\leqslant q$, and $q \not\lesssim p$. Since $p \lesssim' q$, the pair (p,q) arises in $\lesssim'$ from the transitive closure of $\lesssim \setminus \rho$. Thus, there exists a sequence $p = z_0 \lesssim' z_1 \lesssim' \cdots \lesssim' z_r = q$ of elements of P such that $(z_i, z_{i+1}) \in \lesssim \setminus \rho$ for all $i = 0,\dots,r$. The following steps are now needed to obtain the desired sequence.

(1) If, for some $j \in \{1,\dots,r-1\}$, we have $z_{j-1} \lesssim z_j \lesssim z_{j+1}$, and $z_{j+1} \lesssim z_j \lesssim z_{j-1}$, we remove from the sequence the element z_j, obtaining thus a new sequence $\cdots \lesssim' z_{j-1} \lesssim' z_{j+1} \lesssim' \cdots$. Since $z_{j-1} \lesssim z_{j+1}$ and $z_{j+1} \lesssim z_{j-1}$ the properties of the sequence are preserved.

(2) If for some $j \in \{1,\dots,r-1\}$, we have $z_{j-1} \leqslant z_j \leqslant z_{j+1}$, we duplicate the element z_j in the sequence, obtaining thus the sequence $\cdots \lesssim' z_{j-1} \lesssim' z_j \lesssim' z_j \lesssim' z_{j+1} \lesssim' \cdots$.

In any case, by Definition 2.7, we have $p \lesssim_\pi q$.

We conclude that $\lesssim' = \lesssim_\pi$. From Theorem 3.2, we have that π is a regular partition if and only if for any $B_i, B_j \in \pi$, and for all $p \in B_i$, $q \in B_j$,

$$B_i \leqslant B_j \text{ if and only if } p \lesssim_\pi q \text{ if and only if } p \lesssim' q.$$

Therefore, π is a regular partition if and only if $\lesssim = \lesssim' = \mathrm{tr}(\lesssim \setminus \rho)$, and the proof is complete. □

Proposition 4.2 shows that there is a bijection between regular partitions of a poset $(P, \leqslant)$ and quasiorders $\lesssim$ on P extending $\leqslant$ and such that $\lesssim = \mathrm{tr}(\lesssim \setminus \rho)$, where ρ is defined as in the proposition. An example may clarify the bijection at hand.

Example 4.2. Consider the poset P and its partition π used in Example and Figure 4.1. Recall that the quasiorder corresponding to π according to Proposition 4.1 is

$$\lesssim = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(a,c),(b,c),(a,d),(b,d),(d,c)\}.$$

The set $\rho = \{(x,y) \in P \times P \mid x \lesssim y, x \not\leqslant y, y \not\lesssim x\}$ contains only the pair (d,c). Thus, $\mathrm{tr}(\lesssim \setminus \rho) = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a),(a,c),(b,c),(a,d),(b,d)\} \neq \lesssim$, and the partition associated with $\lesssim$ is not a regular partition.

Consider now the poset P' and its partition π' depicted in Figure 4.2. The quasiorder corresponding to π' according to Proposition 4.1 is

$$\lesssim = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(b,c),(a,d),(b,d),(d,b),(d,c)\}.$$

Since $\rho' = \{(d,c)\}$, we have that $\mathrm{tr}(\lesssim \setminus \rho') = \lesssim$ and π' is a regular partition.

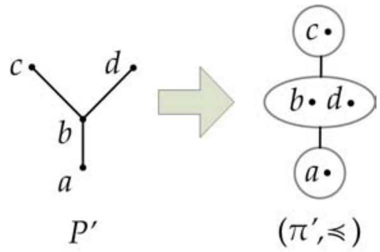


Figure 4.2: *Example 4.2.*

4.2 Partition lattices

In the light of Propositions 4.1 and 4.2, we can think of partitions as quasiorders. More precisely, each quasiorder $\lesssim$ such that $\leqslant \subseteq \lesssim \subseteq P \times P$ defines a unique monotone partition of $(P,\leqslant)$. Moreover, when (and only when) $\lesssim$ satisfies Condition (4.3) in Proposition 4.2, then $\lesssim$ defines a regular partition of $(P,\leqslant)$. Defining an appropriate order between partitions now becomes an easy task, as we can simply consider the set-theoretic inclusion between the associated quasiorders. In this section we obtain the analog of Fact 4.2 for monotone and regular partitions of a poset.

Proposition 4.3. *The collection of monotone partitions of $(P,\leqslant)$ is a lattice when partially ordered by set-theoretic inclusion between the corresponding quasiorders.*

Specifically, let π_1 and π_2 be monotone partitions of $(P,\leqslant)$, and let $\lesssim_1$ and $\lesssim_2$ be the quasiorders corresponding to π_1 and π_2, respectively, as in Proposition 4.1. Then $\pi_1 \wedge_m \pi_2$ and $\pi_1 \vee_m \pi_2$ (the lattice meet and join) are the partitions corresponding to the quasiorders:

$$\lesssim_1 \wedge_m \lesssim_2 = \lesssim_1 \cap \lesssim_2, \quad \lesssim_1 \vee_m \lesssim_2 = \mathrm{tr}(\lesssim_1 \cup \lesssim_2).$$

Proof. We observe that if $\leqslant \subseteq \lesssim_1$ and $\leqslant \subseteq \lesssim_2$, then $\leqslant \subseteq \lesssim_1 \cap \lesssim_2$, and $\leqslant \subseteq \lesssim_1 \cup \lesssim_2$. We also notice that $\lesssim_1 \cap \lesssim_2 = \lesssim_1$ if and only if $\lesssim_1 \subseteq \lesssim_2$. Moreover, both $\wedge_m$ and $\vee_m$ are idempotent, commutative, and associative, because intersection and union are. Finally, the absorption laws

$$\lesssim_1 \wedge_m (\lesssim_1 \vee_m \lesssim_2) = \lesssim_1 \quad \text{and} \quad \lesssim_1 \vee_m (\lesssim_1 \wedge_m \lesssim_2) = \lesssim_1$$

trivially hold. □

Example 4.3. Let us consider, as an example, the poset in Figure 4.3.

In the previous chapter, we have listed all (monotone and regular) partitions of P (see Example and Figure 3.8). Figure 4.4 shows the lattice structure of the set of all monotone partitions of P.

As anticipated, the class of regular partitions also carries a lattice structure.

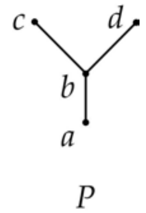


Figure 4.3: *A poset P.*

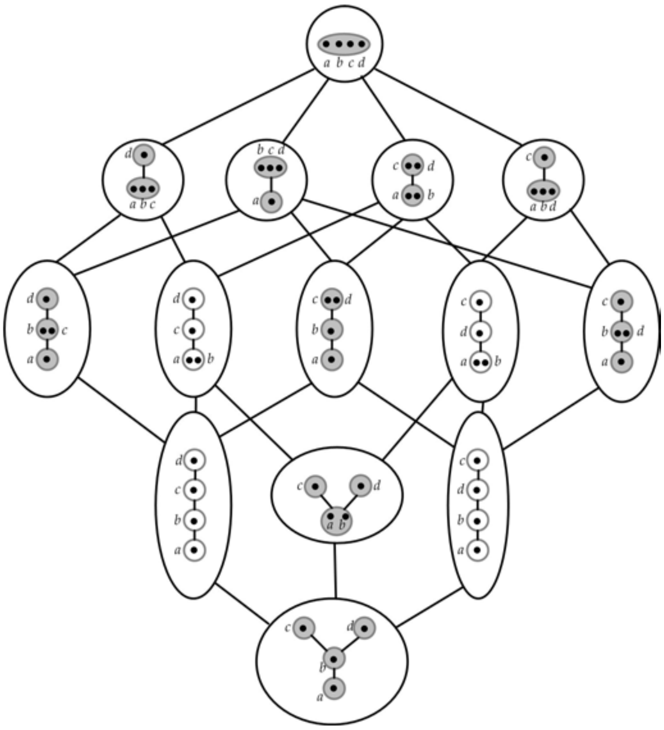


Figure 4.4: *Example 4.3.*

Proposition 4.4. *The collection of regular partitions of $(P, \leqslant)$ is a lattice when partially ordered by set-theoretic inclusion between the corresponding quasiorders.*

Specifically, let π_1 and π_2 be regular partitions of $(P, \leqslant)$, let $\lesssim_1$ and $\lesssim_2$ be the quasiorders corresponding to π_1 and π_2, respectively, and let $\tau = \{(x, y) \in (\lesssim_1 \cap \lesssim_2) \setminus \leqslant \mid y \not\lesssim_1 x \text{ or } y \not\lesssim_2 x\}$ Then $\pi_1 \wedge_r \pi_2$ and $\pi_1 \vee_r \pi_2$ (the lattice meet and join) are the partitions corresponding to the quasiorders:

$$\lesssim_1 \wedge_r \lesssim_2 = \mathrm{tr}((\lesssim_1 \cap \lesssim_2) \setminus \tau), \quad \lesssim_1 \vee_r \lesssim_2 = \mathrm{tr}(\lesssim_1 \cup \lesssim_2).$$

Proof. By construction, $\lesssim_1 \wedge_r \lesssim_2$ defines a regular partition. We now prove that the monotone partition $\lesssim_1 \vee_r \lesssim_2$ defines a regular partition, too. Consider $\lesssim_{12} = \lesssim_1 \cup \lesssim_2$, and let $\tau_{12} = \{(x,y) \in \lesssim_{12} \mid x \nleqslant y,\ y \not\lesssim_{12} x\}$. Suppose $(p,q) \in \tau_{12}$. Say, without loss of generality, $p \lesssim_1 q$. Then, by Proposition 4.2, there exists a sequence $p = z_0 \lesssim_1 z_1 \lesssim_1 \cdots \lesssim_1 z_r = q$ of elements of P such that $(z_i, z_{i+1}) \in \lesssim_1 \setminus \tau_1$ for all $i = 0, \ldots, r$, and $\tau_1 = \{(x,y) \in \lesssim_1 \mid x \nleqslant y,\ y \not\lesssim_1 x\}$. But if $(z_i, z_{i+1}) \notin \tau_1$, then $z_i \leqslant z_{i+1}$, or $z_{i+1} \lesssim_1 z_i$. In both cases $(z_i, z_{i+1}) \notin \tau_{12}$, and thus $(z_i, z_{i+1}) \in \lesssim_{12} \setminus \tau_{12}$ for all i, and $(p,q) \in \mathrm{tr}((\lesssim_1 \cup \lesssim_2) \setminus \tau_{12})$. Hence, $\mathrm{tr}(\lesssim_1 \cup \lesssim_2)$ corresponds to a regular partition.

We can easily check, by the properties of intersection and union, that $\wedge_r$ and $\vee_r$ are idempotent, commutative, associative, and satisfy the absorption laws. It remains to show that $\lesssim_1 \wedge_r \lesssim_2 = \lesssim_1$ if and only if $\lesssim_1 \subseteq \lesssim_2$. Suppose $\lesssim_1 \subseteq \lesssim_2$. Then, $\lesssim_1 \cap \lesssim_2 = \lesssim_1$ and, since $\lesssim_1$ is regular, $\mathrm{tr}((\lesssim_1 \cap \lesssim_2) \setminus \tau) = \mathrm{tr}(\lesssim_1 \setminus \tau) = \lesssim_1$. Suppose now that $\mathrm{tr}((\lesssim_1 \cap \lesssim_2) \setminus \tau) = \lesssim_1$ and let $x \lesssim_1 y$. Then either $(x,y) \in (\lesssim_1 \cap \lesssim_2) \setminus \tau$, or (x,y) is a pair arising from the transitive closure of $(\lesssim_1 \cap \lesssim_2) \setminus \tau$. In any case, since $(\lesssim_1 \cap \lesssim_2) \setminus \tau \subseteq \lesssim_2$ and $\lesssim_2$ is transitive, we have that $x \lesssim_2 y$, proving that if $\mathrm{tr}((\lesssim_1 \cap \lesssim_2) \setminus \tau) = \lesssim_1$, then $\lesssim_1 \subseteq \lesssim_2$. ◻

Example 4.4. The shaded partitions in Figure 4.4 are the regular partitions of the poset P in Figure 4.3. Figure 4.5 depicts the lattice of regular partitions of P.

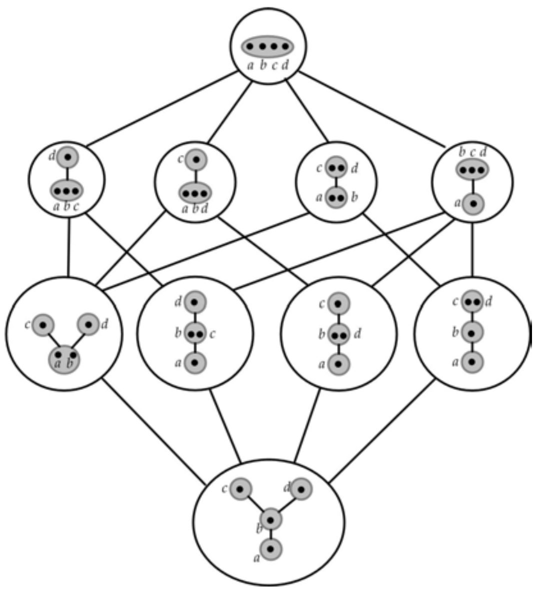


Figure 4.5: *Example 4.4.*

In a set partition lattice, the meet $\wedge$ and the join $\vee$ can be described as follows.

Fact 4.3. *Let π_1 and π_2 be partitions of a set A, and let $\pi_\wedge = \pi_1 \wedge \pi_2$ and $\pi_\vee = \pi_1 \vee \pi_2$. For any $x, y \in A$,*

- *x and y are in the same block of the partition $\pi_\wedge$ if and only if they are in the same block both in π_1 and in π_2,*
- *x and y are in the same block of the partition $\pi_\vee$ if and only if there exists a sequence of blocks $B_1,\ldots,B_k$ in $\pi_1 \cup \pi_2$ such that $x \in B_1$, $y \in B_k$ and for each $i \in \{1,\ldots,k-1\}$, $B_i \cap B_{i+1} \neq \emptyset$.*

As we will see, a similar characterization of meets and joins in the lattices of monotone and regular partitions can be obtained. These characterizations are not so explanatory nor strongly intuitive as their classical counterparts, and that is the reason why we have introduced our lattices as lattices of quasiorders. Nevertheless, they will turn useful in the following. Moreover, it seems important to us to maintain this connection with the classical case where meets and joins are more often defined in terms of blocks than in terms of equivalence relations.

Proposition 4.5. *Let $(\pi_1,\preccurlyeq_1)$ and $(\pi_2,\preccurlyeq_2)$ be monotone partitions of a poset $(P,\leqslant)$, and let $(\pi_{\wedge_m},\preccurlyeq_{\wedge_m}) = \pi_1 \wedge_m \pi_2$ and $(\pi_{\vee_m},\preccurlyeq_{\vee_m}) = \pi_1 \vee_m \pi_2$. Let us indicate with $B_a^{\wedge_m}$, $B_a^{\vee_m}$, B_a^1 and B_a^2 the blocks of $\pi_{\wedge_m}$, $\pi_{\vee_m}$, π_1 and π_2, respectively, containing the element $a \in P$.*

- *The underlying set of $\pi_{\wedge_m}$ is the (set) partition $\pi_1 \wedge \pi_2$. Moreover, for any two blocks B,C of $\pi_{\wedge_m}$, $B \preccurlyeq_{\wedge_m} C$ if and only if there exist $x \in B$ and $y \in C$, such that $B_x^1 \preccurlyeq_1 B_y^1$ and $B_x^2 \preccurlyeq_2 B_y^2$.*
- *For any two elements $x,y \in P$, $B_x^{\vee_m} \preccurlyeq_{\vee_m} B_y^{\vee_m}$ if and only if there exists a sequence of elements of P, $x = z_1,z_2,\ldots,z_{2k} = y$, such that*

$$B^1_{z_1} \preccurlyeq_1 B^1_{z_2}\,,\ \ B^2_{z_2} \preccurlyeq_2 B^2_{z_3}\,,\ldots,\ \ B^2_{z_{2k-2}} \preccurlyeq_2 B^2_{z_{2k-1}}\,,\ \ B^1_{z_{2k-1}} \preccurlyeq_1 B^1_{z_{2k}}\,. \tag{4.4}$$

Proof. Consider the quasiorders $\lesssim_1$ and $\lesssim_2$ associated to π_1 and π_2, respectively, according to Proposition 4.1, and the quasiorder $\lesssim_{\wedge_m}$ associated to $\pi_{\wedge_m}$. By Proposition 4.3 we have that $\lesssim_{\wedge_m} = \lesssim_1 \cap \lesssim_2$. Let $\equiv_1$, $\equiv_2$ and $\equiv_{\wedge_m}$ be the equivalence relations endowed by $\lesssim_1$, $\lesssim_2$ and $\lesssim_{\wedge_m}$, respectively. By Condition (4.2) in Proposition 4.1, we have that, for any $a,b \in P$, $a \equiv_{\wedge_m} b$ if and only if $a \equiv_1 b$ and $a \equiv_2 b$. This implies that a and b are in the same block of $\pi_{\wedge_m}$ if and only if they are in the same block of π_1 and π_2. By Fact 4.3, we have $\pi_{\wedge_m} = \pi_1 \wedge \pi_2$.

Furthermore, by Condition (4.1) in Proposition 4.1, we have that $B \preccurlyeq_{\wedge_m} C$ if and only if $x \lesssim_{\wedge_m} y$ for some $x \in B$, $y \in C$. Since $x \lesssim_{\wedge_m} y$ if and only if $x \lesssim_1 y$ and $x \lesssim_2 y$, if and only if $[x]_{\equiv_1} \lesssim_1 [y]_{\equiv_1}$ and $[x]_{\equiv_2} \lesssim_2 [y]_{\equiv_2}$, the second part of the first statement holds.

To prove the second statement we consider, as before, the quasiorders $\lesssim_1$ and $\lesssim_2$ associated to π_1 and π_2. By Proposition 4.3 we have that $B \preccurlyeq_{\vee_m} C$ if and only if there exist $x \in B$ and $y \in C$ such that the pair (x,y) belong to $\mathrm{tr}(\lesssim_1 \cup \lesssim_2)$. Thus, there exists a sequence of elements of P, $x = z_1,z_2,\ldots,z_{2k} = y$, such that for each $i \in \{1,2k-1\}$, $(z_i,z_{i+1}) \in \lesssim_1$ if i is odd, and $(z_i,z_{i+1}) \in \lesssim_2$ if i is even. Using Condition (4.1) in Proposition 4.1, this fact can be immediately translated in terms of partitions, deriving in this way Condition (4.4). $\square$

Proposition 4.6. *Let* $(\pi_1, \preccurlyeq_1)$ *and* $(\pi_2, \preccurlyeq_2)$ *be regular partitions of a poset* $(P, \preccurlyeq)$*, and let* $(\pi_{\wedge_r}, \preccurlyeq_{\wedge_r}) = \pi_1 \wedge_r \pi_2$ *and* $(\pi_{\vee_r}, \preccurlyeq_{\vee_r}) = \pi_1 \vee_r \pi_2$*.*

- *The underlying set of* $\pi_{\wedge_r}$ *is the (set) partition* $\pi_1 \wedge \pi_2$*. Moreover, for any two blocks* B, C *of* $\pi_{\wedge_r}$*, and for any* $x \in B$*,* $y \in C$*,* $B \preccurlyeq_{\wedge_r} C$ *if and only if* $x \lesssim_{\pi_{\wedge_r}} y$*,*
- *The regular partition* $\pi_{\vee_r}$ *coincides with* $\pi_1 \vee_m \pi_2$*.*

Proof. We note that if two elements $x, y \in P$ are in the same block of π_1 and in the same block of π_2 then neither $(x, y) \in \tau$ nor $(y, x) \in \tau$, where τ is as in Proposition 4.4. Thus, using the same argument as in the proof of Proposition 4.5 we obtain the first part of the first statement. The second part follows directly from Corollary 3.4.

Finally, Propositions 4.3 and 4.4 tell us that given two regular partitions π_1 and π_2, the partition $\pi_1 \vee_m \pi_2$ is regular. □

Some remarks are in order.

Remarks 4.1. Given two regular partitions π_1 and π_2, we notice that the partitions $\pi_1 \wedge_r \pi_2$ and $\pi_1 \wedge_m \pi_2$ are not necessarily the same. In particular, even if their underlying sets coincide, and coincide with the set-theoretic partition $\pi_1 \wedge \pi_2$, the partial orders of $\pi_1 \wedge_r \pi_2$ and $\pi_1 \wedge_m \pi_2$ may differ (see Example 4.5).

By contrast, as we saw in Proposition 4.6, in the monotone partition lattice the join of two regular partitions is a regular partition, that is $\pi_1 \vee_m \pi_2 = \pi_1 \vee_r \pi_2$.

Another consequence of Proposition 4.6 is the following.

Corollary 4.7. *The regular partition lattice is ordered by refinement of the underlying sets of the regular partitions.*

Proof. This is an immediate consequence of Proposition 4.6 and Corollary 3.4. □

Example 4.5. Figures 4.6 and 4.7 show two examples of meets and joins in monotone and regular partition lattices.

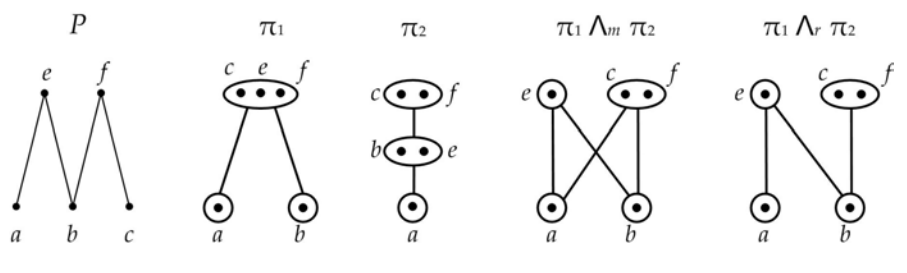


Figure 4.6: *How the* meet *operations work.*

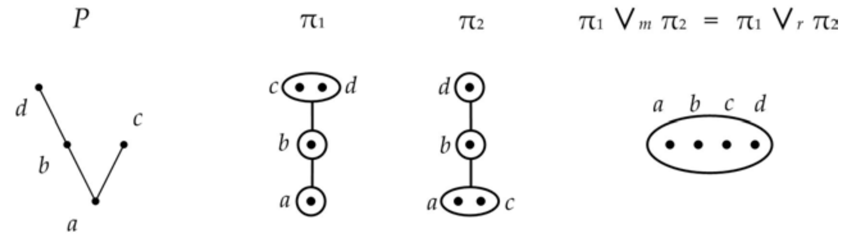


Figure 4.7: *How the* join *operations work.*

4.3 Semilattice embedding of partition lattices

A *sublattice* $(S,\wedge,\vee)$ of a lattice $(L,\wedge,\vee)$ is defined as a subset S of L with the property that for each $a,b \in S$, $a\wedge b, a\vee b \in S$, and the $\wedge$ and the $\vee$ of S are the restrictions of the $\wedge$ and the $\vee$ of L. If S contains $\mathbf{0},\mathbf{1} \in L$, we say that S is a $\{\mathbf{0},\mathbf{1}\}$-*sublattice* of L. If S is a subset of L such that for each $a,b \in S$, $a\wedge b \in S$, and the $\wedge$ of S is the restriction of the $\wedge$ of L, we say that S is a *meet-subsemilattice* of L. Other notions that will be used in the following are that of *join-subsemilattice*, $\{\mathbf{0},\mathbf{1}\}$-*meet-subsemilattice*, and $\{\mathbf{0},\mathbf{1}\}$-*join-subsemilattice*.

The following propositions establish interesting relationships among partition lattices, monotone partition lattices and regular partition lattices.

Proposition 4.8. *The regular partition lattice of a poset* $(P,\leqslant)$ *is (isomorphic to) a* $\{\mathbf{0},\mathbf{1}\}$-*meet-subsemilattice of the partition lattice of* P.

Proof. Let $(\Pi_r,\wedge_r,\vee_r,\mathbf{0}_r,\mathbf{1}_r)$ be the regular partition lattice of $(P,\leqslant)$, and let $(\Pi,\wedge,\vee,\mathbf{0},\mathbf{1})$ be the partition lattice of P. Consider the map $f:(\Pi_r,\wedge_r,\vee_r,\mathbf{0}_r,\mathbf{1}_r)\to(\Pi,\wedge,\vee,\mathbf{0},\mathbf{1})$ defined by

$$(\pi,\leqslant)\in\Pi_r \mapsto \pi\in\Pi\,.$$

We claim that f is a $\{\mathbf{0},\mathbf{1}\}$-meet-subsemilattice embedding, *i.e.*, an injection that preserves meets, top and bottom.

By Corollary 3.4 f is injective. It is an exercise to verify that $\mathbf{0}$ and $\mathbf{1}$ admit an extension to a regular partition of P, according to Corollary 3.4. Moreover, by Proposition 4.4, $\mathbf{0}_r$ is the regular partition corresponding to the quasiorder obtained by the set-theoretic union of the identity relation on P and the order $\leqslant$ of P, and $\mathbf{1}_r$ is the regular partition corresponding to the universal relation on P. Thus, $f(\mathbf{0}_r)=\mathbf{0}$, and $f(\mathbf{1}_r)=\mathbf{1}$. Finally, by Proposition 4.6, for each $\pi_1,\pi_2\in\Pi_r$, $f(\pi_1\wedge_r\pi_2)=f(\pi_1)\wedge f(\pi_2)$. □

Proposition 4.9. *The regular partition lattice of a poset* $(P,\leqslant)$ *is a* $\{\mathbf{0},\mathbf{1}\}$-*join-subsemilattice of the monotone partition lattice of* $(P,\leqslant)$.

Proof. Let $(\Pi_m,\wedge_m,\vee_m,\mathbf{0}_m,\mathbf{1}_m)$ be the regular partition lattice of the poset $(P,\leqslant)$, and let $(\Pi_r,\wedge_r,\vee_r,\mathbf{0}_r,\mathbf{1}_r)$ be its regular partition lattice. Trivially, we obtain that $\Pi_r\subseteq\Pi_m$, $\mathbf{0}_m=\mathbf{0}_r$, $\mathbf{1}_m=\mathbf{1}_r$. By Proposition 4.6, for any $\pi_1,\pi_2\in\Pi_r$, $\pi_1\vee_r\pi_2=\pi_1\vee_m\pi_2$. □

Example 4.6. Figure 4.8 shows the regular partition lattice depicted in Figure 4.5 as a $\{\mathbf{0},\mathbf{1}\}$-meet-subsemilattice of the set $\{a,b,c,d\}$. Partitions corresponding to regular partitions are grey. A label like $a/b/cd$ is to be interpreted as the partition $\{\{a\},\{b\},\{c,d\}\}$. On the other hand, an example for Proposition 4.9 is provided by Figure 4.4.

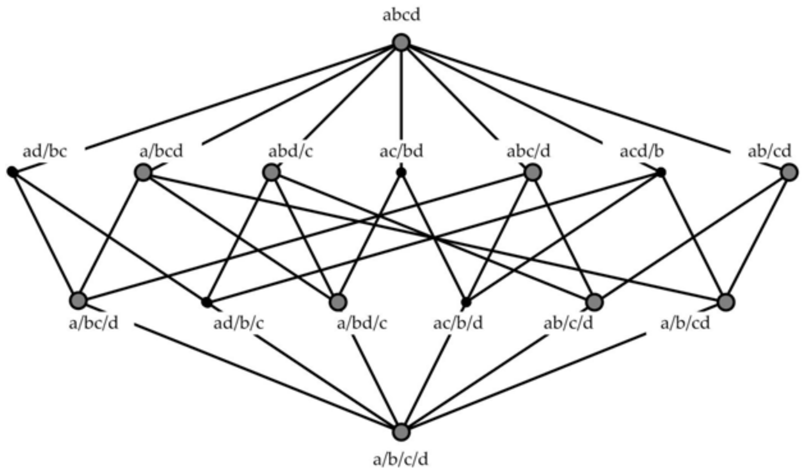


Figure 4.8: *Regular partition lattices as $\{\mathbf{0},\mathbf{1}\}$-meet-subsemilattices of partition lattices.*

4.4 The lattice of regular partitions is ranked

Many important properties of the (set) partition lattice have been investigated, and a full characterization of this lattice is known – see the bibliographic notes at the end of this chapter. A systematic comparison between the partition lattice and the monotone and regular partition lattices would surely be of interest. However, such a treatment will have to await further research. Here, we limit ourselves to a short discussion on ranked and geometric lattices. It is well known that partition lattices are both ranked and geometric.

To obtain our first result we prepare two lemmas.

Lemma 4.1. *Let $(\pi,\preccurlyeq)$ be a regular partition of $(P,\leqslant)$, and let $B_1,B_2 \in \pi$, with $B_1 \neq B_2$. Let $\overline{\pi}$ be the (set) partition obtained from π by the union of B_1 and B_2, in symbols $\overline{\pi} = (\pi \setminus \{B_1,B_2\}) \cup (B_1 \cup B_2)$. Then, the following hold.*

1. *If B_1 covers B_2 in π, or viceversa, then $\overline{\pi}$ extends to a regular partition of π.*
2. *If B_1 and B_2 are incomparable in π, then $\overline{\pi}$ extends to a regular partition of π.*
3. *B_1 and B_2 are comparable, but neither one cover the other, if and only if $\overline{\pi}$ does not extend to a regular partition of π.*

Proof. Let $(\pi, \preccurlyeq)$ be a regular partition of $(P, \preccurlyeq)$, and let $B_1, B_2 \in \pi$, with $B_1 \neq B_2$. Let $\overline{\pi}$ be the (set) partition obtained from π by the union of B_1 and B_2.

1. Suppose, without loss of generality, that B_2 covers B_1. Let $\lesssim$ be the quasiorder associated to π, according to Proposition 4.2. We construct a quasiorder by setting:

$$\overline{\lesssim} = \mathrm{tr}(\lesssim \cup \{(y,x) \in P \times P \mid x \in B_1, y \in B_2\}).$$

We want to show that the (set) partition induced by $\overline{\lesssim}$ coincide with $\overline{\pi}$. In other words, we need to prove that for any $x, y \in P$

$$x \mathrel{\overline{\lesssim}} y,\ y \mathrel{\overline{\lesssim}} x,\ y \not\lesssim x \text{ if and only if } x \in B_1,\ y \in B_2\,. \tag{4.5}$$

If $x \in B_1$, $y \in B_2$, by the definition of $\overline{\lesssim}$, we have $y \mathrel{\overline{\lesssim}} x$. Since $B_1 \preccurlyeq B_2$ and $B_1 \neq B_2$, by Condition (4.1) in Proposition 4.1, we have that $x \lesssim y$, and $y \not\lesssim x$, and one side of (4.5) is proved.

Suppose now that for some $z, w \in P$ such that $z \mathrel{\overline{\lesssim}} w$, $w \mathrel{\overline{\lesssim}} z$, $w \not\lesssim z$, we have $z \in B_z \neq B_1$, or $w \in B_w \neq B_2$. Then, $w \not\lesssim z$ arise in $\overline{\pi}$ by transitive closure, that is, there exist a pair (x, y), with $x \in B_1$ and $y \in B_2$, such that $w \lesssim y$ and $x \lesssim w$. By Condition (4.1) in Proposition 4.1 we have $B_z \preccurlyeq B_2$ and $B_1 \preccurlyeq B_w$. There are now two cases for the pair (z, w).

(Case 1). If $(z, w) \in \lesssim$, then $x \lesssim z \lesssim w \lesssim y$. By Condition (4.1) in Proposition 4.1 we have $B_1 \lesssim B_z \lesssim B_w \lesssim B_2$. Since one of $B_z \neq B_1$, $B_w \neq B_2$ hold, and since $B_z = B_2$, $B_w = B_1$ cannot hold, for otherwise we would have $w \lesssim z$, we contradict our hypothesis that B_2 covers B_1.

(Case 2). If $(z, w) \notin \lesssim$, then (z, w) arise in $\overline{\pi}$ by transitive closure, that is $z \lesssim y$ and $x \lesssim w$. Thus, we have $x \lesssim w \lesssim y$ and $x \lesssim z \lesssim w$. By Condition (4.1) in Proposition 4.1 we have $B_1 \lesssim B_z \lesssim B_2$ and $B_1 \lesssim B_w \lesssim B_2$. Since one of $B_z \neq B_1$, $B_w \neq B_2$ hold, and since $B_z = B_2$, $B_w = B_1$ cannot hold, for otherwise we would have $w \lesssim z$, we contradict again our hypothesis that B_2 covers B_1.

Observing that $\overline{\lesssim}$ extends $\lesssim$ and that $\overline{\lesssim}$ adds to $\lesssim$ only pairs (y, x) such that $x \lesssim y$ or pairs obtainable by transitive closure, we have that the hypothesis of Proposition 4.2 are satisfied and $\overline{\lesssim}$ is the quasiorder associated to a regular partition of P. Since $\overline{\pi}$ is the (set) partition induced by $\overline{\lesssim}$, $\overline{\pi}$ extends to the regular partition associated to $\overline{\lesssim}$.

2. Suppose that B_1 and B_2 are incomparable in π. Let $\lesssim$ be the quasiorder associated to π, according to Proposition 4.2. We construct a quasiorder by setting:

$$\overline{\lesssim} = \mathrm{tr}(\lesssim \cup \{(x,y) \in P \times P \mid x \in B_1, y \in B_2\} \cup \{(y,x) \in P \times P \mid x \in B_1, y \in B_2\}).$$

We want to show that the (set) partition induced by $\overline{\lesssim}$ coincide with $\overline{\pi}$. In other words, we need to prove that the pairs of elements $x, y \in P$ such that $(x, y), (y, x) \in \overline{\lesssim}$, are such that $x \lesssim y$ and $y \lesssim x$, or such that $x \in B_1$ and $y \in B_2$. Suppose there exist $(z, w), (w, z) \in \overline{\lesssim}$ not satisfying such conditions. Then, one of $(z, w), (w, z)$, or both, arise in $\overline{\pi}$ by transitive closure. Say, without loss of generality, (z, w) is such a pair. Then, there exist $(x, y) \notin \lesssim$, $x \in B_1$ and $y \in B_2$ (the case $y \in B_1$ and $x \in B_2$ is analogous), such that $z \lesssim x$ and $y \lesssim w$. We have now two cases.

(Case 1). If $(w,z) \in \lesssim$ then $y \lesssim w \lesssim z \lesssim x$. By Condition (4.1) in Proposition 4.1, $y \lesssim x$ implies $B_2 \preccurlyeq B_1$, contradicting the hypothesis that B_1 and B_2 are incomparable.

(Case 2). If $(w,z) \notin \lesssim$ then (w,z) arise in $\overline{\pi}$ by transitive closure and there exists a pair $(x',y') \notin \lesssim$, with $x' \in B_1, y' \in B_2$ or $x' \in B_2, y' \in B_1$, such that $w \lesssim x'$ and $y' \lesssim z$. Thus, we have $y \lesssim w \lesssim x'$ and $y' \lesssim z \lesssim x$. Again, we make use of the Condition (4.1) in Proposition 4.1. If $x' \in B_1$ and $y' \in B_2$ we obtain $B_2 \preccurlyeq B_1$, contradicting the hypothesis that B_1 and B_2 are incomparable. If $x' \in B_2$ and $y' \in B_1$ we obtain $w \in B_2$ and $z \in B_1$, contradicting the fact that (z,w) and (w,z) arise from the transitive closure.

We observe now that $\overline{\lesssim}$ extends $\lesssim$ and that $\overline{\lesssim}$ adds to $\lesssim$ only couple of pairs $(x,y),(y,x)$ or pairs obtainable by transitive closure. Thus, the hypothesis of Proposition 4.2 are satisfied and $\overline{\lesssim}$ is the quasiorder associated to a regular partition of P. Since $\overline{\pi}$ is the (set) partition induced by $\overline{\lesssim}$, $\overline{\pi}$ extends to the regular partition associated to $\overline{\lesssim}$.

3. Suppose $B_1 \preccurlyeq B_2$ and suppose that there exists $B_3 \in \pi$, $B_3 \neq B_1 \neq B_2$, such that $B_1 \preccurlyeq B_3 \preccurlyeq B_2$. Since π is a regular partition, by Theorem 3.2 we have that for all $x \in B_1$, $y \in B_2$, $z \in B_3$, $x \lesssim_\pi z \lesssim_\pi y$ (recall that $\lesssim_\pi$ denote the blockwise quasiorder induced by π.) By Definition 2.7 we immediately obtain $x \lesssim_{\overline{\pi}} z \lesssim_{\overline{\pi}} y$. Since x and y are in the same block in $\overline{\pi}$, we also obtain $x \lesssim_{\overline{\pi}} z \lesssim_{\overline{\pi}} y \lesssim_{\overline{\pi}} x$. Thus, we have $x \lesssim_{\overline{\pi}} z$ and $z \lesssim_{\overline{\pi}} x$. By Corollary 3.4 the partition $\overline{\pi}$ does not extend to a regular partition.

The other side of this statement follows directly by the first and the second statement of the lemma. □

Lemma 4.2. *Let* $(\Pi, \preccurlyeq_\Pi)$ *be the regular partition lattice of* $(P, \preccurlyeq)$*, and consider the regular partitions* $(\pi_1, \preccurlyeq_1), (\pi_2, \preccurlyeq_2) \in \Pi$. *If* $\pi_1 \preccurlyeq_\Pi \pi_2$ *then every chain of regular partitions* $\pi_1 = \sigma_1 \lhd_\Pi \sigma_2 \lhd_\Pi \cdots \lhd_\Pi \sigma_k = \pi_2$ *is such that for all* $i \in \{1,\dots,k-1\}$, σ_{i+1} *is obtained from* σ_i *by the union of two blocks* $B, C \in \sigma_i$ *such that one of the following conditions hold.*

1. *B and C are incomparable in* σ_i.

2. *B covers C in* σ_i.

Proof. Consider a regular partition $(\pi = \{B_1,\dots,B_k\}, \preccurlyeq) \in \Pi$. Suppose that the covering set of π contains a partition σ such that $|\sigma| \geq |\pi| + 2$. Thus, there exists a block $C \in \sigma$ and a set of index $K \subseteq \{1,\dots,k\}$, $|K| \geq 2$, such that $C = \bigcup_{i \in K} B_i$. We have three cases.

(Case 1). There exist $j,h \in K$ such that B_h covers B_j in π. By Lemma 4.1 the partition $\overline{\pi}$ obtained from π by the union of B_j and B_h extends to a regular partition. Since $\pi \preccurlyeq_\Pi \overline{\pi} \preccurlyeq_\Pi \sigma$, and $\overline{\pi} \neq \sigma$, we contradict the fact that σ covers π.

(Case 2). There exist $j,h \in K$ such that B_h and B_j are incomparable in π. By Lemma 4.1 the partition $\overline{\pi}$ obtained from π by the union of B_j and B_h extends to a regular partition. Since $\pi \preccurlyeq_\Pi \overline{\pi} \preccurlyeq_\Pi \sigma$, and $\overline{\pi} \neq \sigma$, we contradict the fact that σ covers π.

(Case 3). For any $j,h \in K$, B_h and B_j are comparable, but not in the covering relation of π. Say, without loss of generality, $B_j \preccurlyeq B_h$, and suppose $B_m \in \pi$ is such that $B_m \neq B_j \neq B_h$ and $B_j \preccurlyeq B_m \preccurlyeq B_h$. Since π is a regular partition, by Theorem 3.2 we have that for all $x \in B_j$, $y \in B_m$, $z \in B_h$, $x \lesssim_\pi y \lesssim_\pi z$. By Definition 2.7 we immediately obtain $x \lesssim_\sigma y \lesssim_\sigma z$. Since x

and z are in the same block in σ, we also obtain $x \lesssim_\sigma y \lesssim_\sigma z \lesssim_\sigma x$. Thus, we have $x \lesssim_\sigma y$ and $y \lesssim_\sigma x$. By Corollary 3.4, since x and y does not belong to the same block of σ, we contradict the fact that σ is a regular partition.

We have so proved that the covering set of $\pi \in \Pi$ contains only partitions σ such that $|\sigma| = |\pi| + 1$. By Lemma 4.1, our result follows. □

We are in a position to prove:

Theorem 4.10. *The regular partition lattice* $(\Pi, \leqslant_\Pi)$ *of a poset* $(P, \leqslant)$ *is ranked by the number of blocks of its elements. Specifically, the rank of an element* $\pi \in \Pi$ *is given by* $|P| - |\pi|$.

Proof. We consider the function $r : \Pi \to \{0, 1, \ldots\}$, defined by $r(\pi) = |P| - |\pi|$. We need to show that r is a rank function on Π. That is, we need to show that the bottom element of $(\Pi, \leqslant_\Pi)$ takes value 0 under r, and that if $\pi_1, \pi_2 \in \Pi$ are such that π_2 covers π_1, then $r(\pi_2) = r(\pi_1) + 1$. The first fact is obvious – just observe that the bottom partition has exactly $|P|$ blocks. The second fact follows from Lemma 4.2. □

Theorem 4.10 does not hold for the monotone partition lattice, as shown by the following example.

Example 4.7. We exhibit a poset whose monotone partition lattice is not ranked. Consider the poset P in Figure 4.9. Figure 4.10 shows all monotone partitions of P. Figure 4.11

Figure 4.9: *A poset with an unranked monotone partition lattice.*

shows the monotone partition lattice of P. Labels denote the positions of the partitions as represented in Figure 4.10, counted from left to right, and from top to bottom.

Partition lattices are known to be an important family of geometric lattices. One may wonder whether Theorem 4.10 can be strengthed to show that regular partition lattices are geometric.[1] In closing this section, we show that it cannot. In fact, the following example shows that in general they are not even semimodular.

Example 4.8. We consider the poset P in Figure 4.12. Figure 4.13 displays all regular partitions of P. Figure 4.14 shows the regular partition lattice of P. Labels denote the

[1]Monotone partition lattices certainly are not geometric in general, as need not be ranked – see Example 4.7.

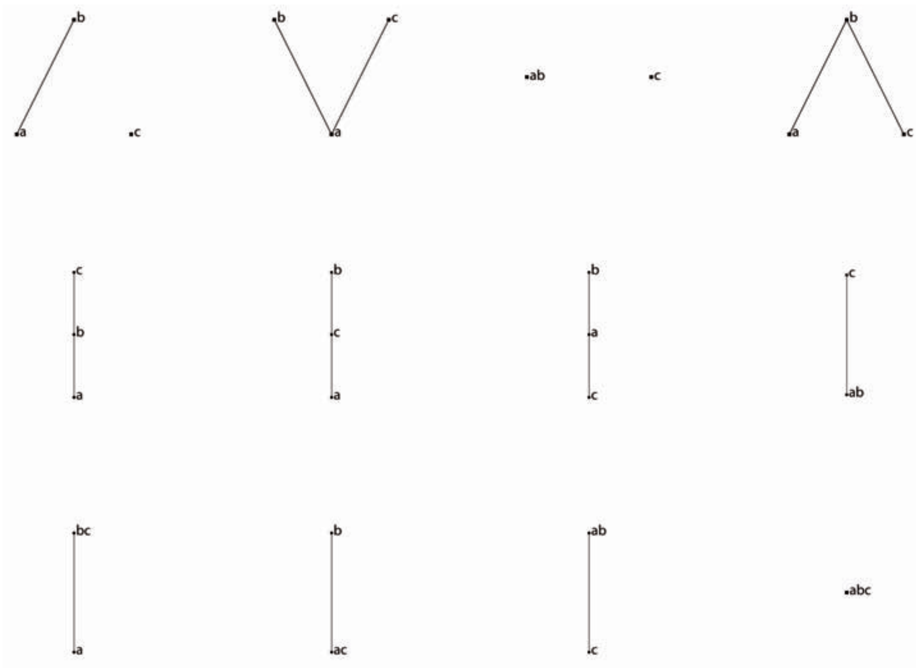


Figure 4.10: *Monotone partitions which form an unranked lattice.*

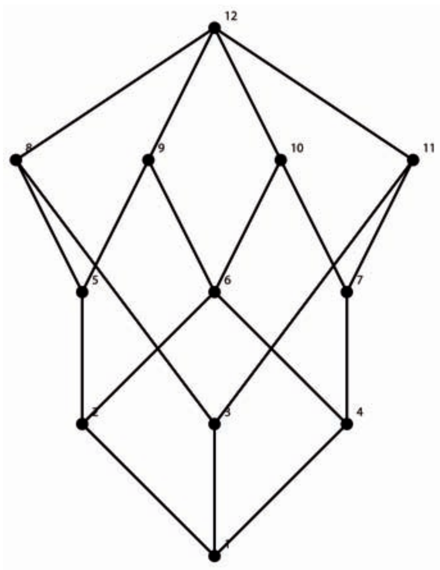


Figure 4.11: *An unranked monotone partition lattice.*

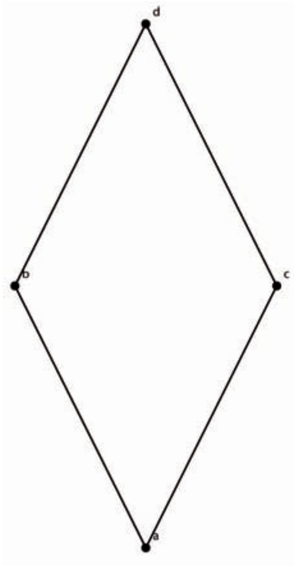


Figure 4.12: *A poset with a non-geometric regular partition lattice.*

positions of the partitions as represented in Figure 4.13, counted from left to right, and from top to bottom. We can immediately observe that 4 and 5 cover $4 \wedge 5$, but $4 \vee 5$ does not cover 4 and does not cover 5. Then the regular partition lattice of P is not semimodular. Therefore, it is not a geometric.

4.5 Bibliographic notes

An extensive analysis of the theory of equivalence relations can be found in [Ore42]. In particular we refer to this paper for Facts 4.1, 4.2, 4.3. Results on partition lattices can also be found, *e.g*, in [Grä98], [BS81], [Sta97]. The characterization of partition lattices mentioned in this chapter can be found in [Sac61].

Many results are also known about the lattice of quasiorders, and, thanks to Propositions 4.1 and 4.3, almost all of them can be related to our investigation of the lattice of monotone partitions. Most of these papers are concerned with the duality that we will treat in the next chapter. Here, it seems appropriate to cite [Ric98], where some important results on lattices of quasiorders can be found, and [ER95], where the structure of intervals in the lattice of all quasiorders is analyzed. In none of these papers there appear notions related to the regular partition lattice.

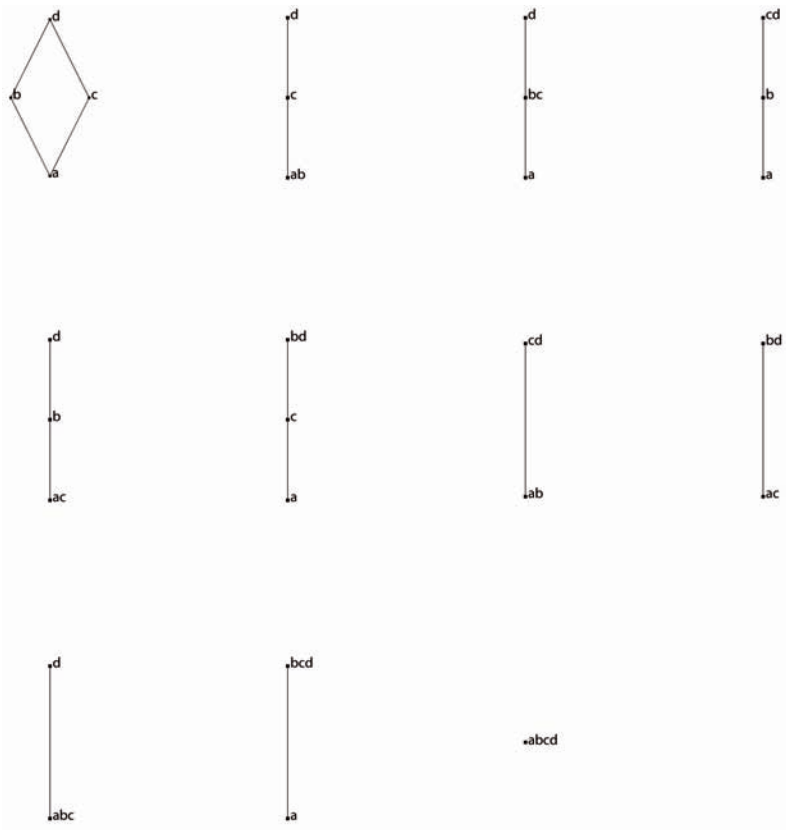


Figure 4.13: *Regular partitions which form a non-geometric lattice.*

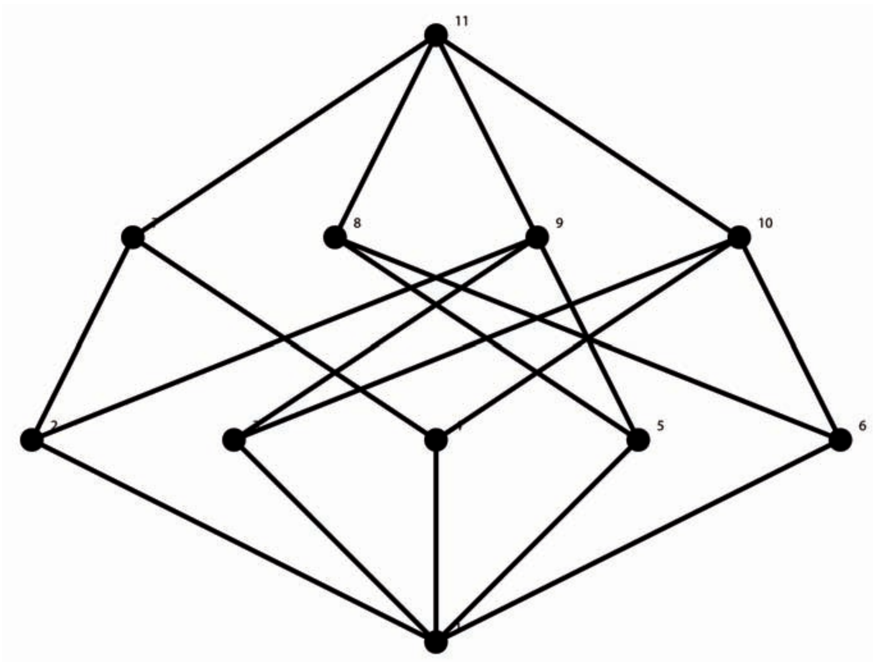


Figure 4.14: *A non-geometric regular partition lattice.*

Chapter 5

Birkhoff duality

Important ideas come in pairs.

Michael Arbib, Ernest Manes

5.1 Equivalence of categories

We turn our focus on categories, to illustrate a fundamental duality for the category `Poset`. In this section, we introduce the notions of equivalence and duality of categories. They are basic for what follows.

To distinguish objects and morphisms in different categories, we call $\mathcal{K}$-*objects* the objects of a category $\mathcal{K}$, and $\mathcal{K}$-*morphisms* its morphisms. In this section, we consider two categories $\mathcal{K}$ and $\mathcal{L}$.

In Chapter 2 we have introduced the notions of category, objects, morphisms, etc. We take now a more global viewpoint and consider the categories themselves as structured objects. The morphisms between categories which preserve their structure are called *functors*.

Definition 5.1. A *functor* F from $\mathcal{K}$ to $\mathcal{L}$ is a correspondence that assigns to each $\mathcal{K}$-object A an $\mathcal{L}$-object $F(A)$, and to each $\mathcal{K}$-morphism $A \xrightarrow{f} B$ an $\mathcal{L}$-morphism $F(A) \xrightarrow{F(f)} F(B)$ in such a way that

(1) F preserves composition, *i.e.*, $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined,

(2) F preserves identities, *i.e.*, $F(id_A) = id_{F(A)}$ for each $\mathcal{K}$-object A.

A functor F from $\mathcal{K}$ to $\mathcal{L}$ will be denoted by $F : \mathcal{K} \to \mathcal{L}$ or $\mathcal{K} \xrightarrow{F} \mathcal{L}$.

We sometimes make use of the notations FA and Ff rather than $F(A)$ and $F(f)$, and sometimes denote the action on both objects and morphisms by

$$F(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FB.$$

Functors are also called *covariant functors*. Note that a functor $F : \mathcal{K} \to \mathcal{L}$ is technically a family of correspondences, one from the class of $\mathcal{K}$-objects to the class of $\mathcal{L}$-objects, and one from $\mathcal{K}$-morphisms to $\mathcal{L}$-morphisms.

A special functor is the *identity functor* $id_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$, defined by

$$id_{\mathcal{K}}(A \xrightarrow{f} B) = A \xrightarrow{f} B.$$

For some categories, there is a notion of *forgetful functor*. For instance, in `Poset` a forgetful functor is defined as the functor $U : \mathtt{Poset} \to \mathtt{Set}$, where $U(P)$ is the underlying set of P, and $U(f)$ is the underlying function of the order-preserving map f. Thus, U "forgets" the partial order of a poset.

Each functor $F : \mathcal{K} \to \mathcal{L}$ *preserves isomorphism*, *i.e.*, whenever $A \xrightarrow{h} A'$ is a $\mathcal{K}$-isomorphism, then Fh is an $\mathcal{L}$-isomorphism. By contrast, note that functors do not *reflect* isomorphisms, meaning that if Fh is an isomorphism, then h needs not to be an isomorphism. Further observe that the *composite* $G \circ F : \mathcal{K} \to \mathcal{M}$ of the functors $F : \mathcal{K} \to \mathcal{L}$ and $G : \mathcal{L} \to \mathcal{M}$, defined by

$$(G \circ F)(A \xrightarrow{f} A') = G(FA) \xrightarrow{G(Ff)} G(FA')$$

is again a functor.

Definition 5.2. A functor $F : \mathcal{K} \to \mathcal{L}$ is called an *isomorphism* provided that there is a functor $G : \mathcal{L} \to \mathcal{K}$ such that $G \circ F = id_{\mathcal{K}}$ and $F \circ G = id_{\mathcal{L}}$. If there is an isomorphism $F : \mathcal{K} \to \mathcal{L}$, we say that $\mathcal{K}$ and $\mathcal{L}$ are *isomorphic categories*.

Isomorphic categories are considered to be essentially the same. In category theory, the term "equivalent categories" formally refers to a weaker relation then just isomorphism. To introduce a formal definition of equivalence of categories, we need to define some other special functors.

Definition 5.3. Let $F : \mathcal{K} \to \mathcal{L}$ be a functor.

(1) F is called an *embedding* provided that is injective on morphisms.

(2) F is called *faithful* provided that the restrictions $F : \mathcal{K}(A, A') \to \mathcal{L}(FA, FA')$ are injective, for any two $\mathcal{K}$-objects A, A'.

(3) F is called *full* provided that all the restrictions $F : \mathcal{K}(A, A') \to \mathcal{L}(FA, FA')$ are surjective, for any two $\mathcal{K}$-objects A, A'.

(4) F is called *essentially surjective* provided that for any $\mathcal{L}$-object B there exist some $\mathcal{K}$-object A such that $F(A)$ is isomorphic to B.

Remark 5.1. Note that a functor is an embedding if and only if it is faithful and injective *on objects*, and that a functor is an isomorphism if and only if it is full, faithful, and bijective on objects.

We are now in a position to define equivalence of categories.

Definition 5.4. A functor $F : \mathcal{K} \to \mathcal{L}$ is called an *equivalence* provided that it is full, faithful, and essentially surjective. Categories $\mathcal{K}$ and $\mathcal{L}$ are called *equivalent* provided that there is an equivalence from $\mathcal{K}$ to $\mathcal{L}$.

Clearly, each isomorphism between categories is an equivalence. Although for some categories equivalences coincide with isomorphisms, this is not true in general, and some categories can be equivalent without being isomorphic. The following fact holds for equivalences.

Fact 5.1. *If $\mathcal{K} \xrightarrow{F} \mathcal{L}$ is an equivalence, then there exists an equivalence $\mathcal{L} \xrightarrow{G} \mathcal{K}$. If $\mathcal{K} \xrightarrow{F} \mathcal{L}$ and $\mathcal{L} \xrightarrow{H} \mathcal{M}$ are equivalences, then so is $\mathcal{K} \xrightarrow{H \circ F} \mathcal{M}$.*

The concept of equivalence is especially useful when the concept of *duality* is involved. There are numerous cases of pairs of categories where each category is equivalent to the dual of the other, and this will be our case. We thus need to formalize the notion of duality.

Definition 5.5. For any category $\mathcal{K}$ the *dual category of $\mathcal{K}$* is the category $\mathcal{K}^{op}$, where $\mathcal{K}^{op}(A, B) = \mathcal{K}(B, A)$ and $f \circ^{op} g = g \circ f$.

The dual category of $\mathcal{K}$ is sometimes called the *opposite category of $\mathcal{K}$*. We note that $\mathcal{K}$ and $\mathcal{K}^{op}$ have the same objects, while the direction of morphisms is reversed. Because of the way dual categories are defined, every statement concerning an object A in $\mathcal{K}$ can be translated into a logically equivalent statement concerning the object A in $\mathcal{K}^{op}$. In the same way, any property about morphisms gives rise to a dual property about morphisms. For any category $\mathcal{K}$ and property P the following hold.

(1) $(\mathcal{K}^{op})^{op} = \mathcal{K}$.

(2) $P^{op}(\mathcal{K})$ holds if and only if $P(\mathcal{K}^{op})$ holds.

Often the dual concept P^{op} is denoted by $co - P$. For instance, limits in $\mathcal{K}$ – which we shall not define here – correspond to colimits in $\mathcal{K}^{op}$. A concept is called *self-dual* if $P = P^{op}$. For example, the statement "f is an isomorphism" is self dual, *i.e.*, f is an isomorphism in $\mathcal{K}$ if and only if f is an isomorphism in $\mathcal{K}^{op}$.

We can now define the notion of *dual equivalence*.

Definition 5.6. Categories $\mathcal{K}$ and $\mathcal{L}$ are called *dually equivalent* provided that $\mathcal{K}^{op}$ and $\mathcal{L}$ are equivalent.

A *contravariant functor* from $\mathcal{K}$ to $\mathcal{L}$ is a functor from $\mathcal{K}^{op}$ to $\mathcal{L}$. Thus, $\mathcal{K}$ and $\mathcal{L}$ are dually equivalent if and only if there exists a contravariant functor from $\mathcal{K}$ to $\mathcal{L}$ which is an equivalence.

In the following example, we illustrate a celebrated duality of categories. Another important duality, for the category `Poset`, will be presented in the next section.

Example 5.1. `Set` is dually equivalent to the category of complete atomic boolean algebras and complete Boolean homomorphisms. An equivalence $\mathcal{P}$ can be obtained by associating with each set its power-set, considered as a complete atomic Boolean algebra. If $f : A \to B$ is a function, then the equivalence $\mathcal{P}$ associate with f the morphism $\mathcal{P} = f^* : \mathcal{P}(B) \to \mathcal{P}(A)$ defined by

$$f^*(S) = f^{-1}(S),$$

for each $S \subseteq B$. Figure 5.1 depicts the action of the functor $\mathcal{P}$.

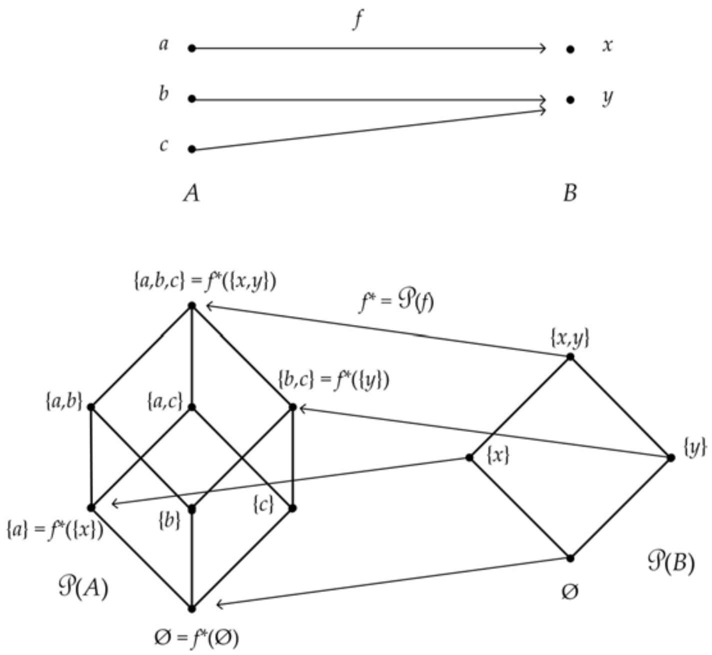


Figure 5.1: *Duality for the category* `Set`.

5.2 Duality for the category `Poset`

Let P be a poset, and let $\mathcal{O}(P)$ be the collection of all downsets of P. The set $\mathcal{O}(P)$ ordered by inclusion is itself a partially ordered set. In fact, it is immediately seen that $\mathcal{O}(P)$ is a distributive lattice. We call $\mathcal{O}(P)$ the *dual* of P. Let now P, Q be posets, and consider an order-preserving map $f : P \to Q$. Let $D \in \mathcal{O}(Q)$. We observe that $f^{-1}(D) \in \mathcal{O}(P)$ if and only if f is order-preserving. We can thus define the map $f^* : \mathcal{O}(Q) \to \mathcal{O}(P)$, by setting, for each $D \in \mathcal{O}(Q)$

$$f^*(D) = f^{-1}(D). \tag{5.1}$$

We say that $f^* = \mathcal{O}(f)$ is the *dual map* of f. It is possible to show that f^* preserve the structure of bounded distributive lattice.

Consider now the category `DL` of (finite, as all the objects studied in this thesis) bounded distributive lattices and lattice homomorphisms preserving top and bottom. The map $\mathcal{O}$ is a functor from `Poset` to $\mathtt{DL}^{op}$. Moreover, one can check that $\mathcal{O}$ preserves composition and identities, and that $\mathcal{O}$ is faithful, full, and essentially surjective. Thus, $\mathcal{O}$ is an equivalence, and the categories `Poset` and `DL` are dually equivalent.

Example 5.2. Figure 5.2 shows the image of an order-preserving map $f : P \to Q$ via $\mathcal{O}$. The morphism f from the poset P to the poset Q is shown in the top half of the picture. In the bottom half, one can see the dual of f, that is the `DL`-morphism f^* from the distributive lattice $\mathcal{O}(Q)$ to the distributive lattice $\mathcal{O}(P)$.

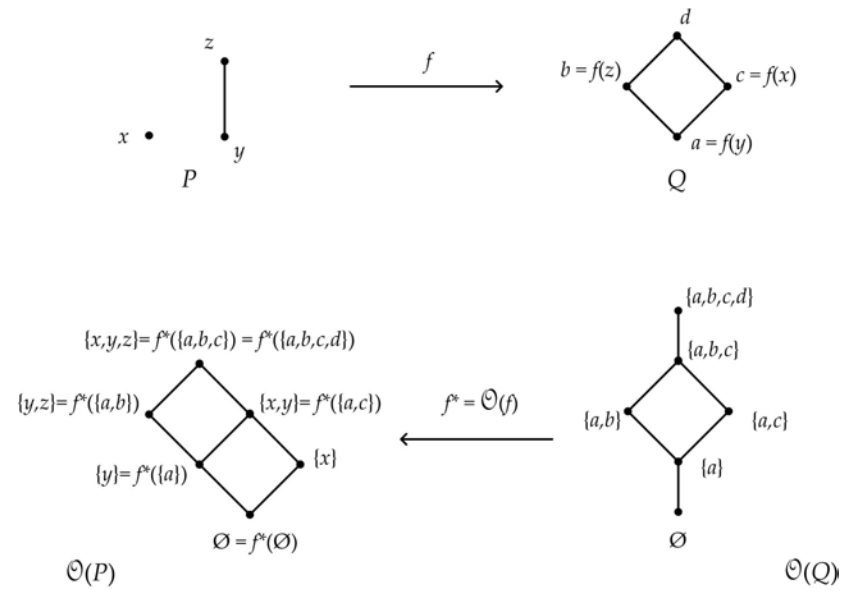


Figure 5.2: *Duality between* `Poset` *and* `DL` *via* $\mathcal{O}$.

By Fact 5.1, there exists an equivalence from `DL` to $\mathtt{Poset}^{op}$. We present here two such equivalences. To introduce the first one, we need some more definitions.

Definition 5.7. Let L be a lattice. A nonempty subset $G \subseteq L$ is called a *filter* if

(i) $x, y \in G$ imply $x \wedge y \in G$,

(ii) $x \in L$, $y \in G$ and $y \leqslant x$ imply $x \in G$.

A filter is called *proper* if it does not coincide with L.

We observe that given a lattice L and an element $x \in L$, the set $\uparrow x = \{y \in L \mid x \leqslant y\}$ is a filter. We call $\uparrow x$ the *principal filter* generated by x. It is possible to verify that in a finite lattice every filter is principal.

Definition 5.8. Let L be a lattice, and $G \subseteq L$ a proper filter of L. Then G is said to be *prime* if $x, y \in L$ and $x \vee y \in G$ imply $x \in G$ or $y \in G$.

We define now the functor Spec which associates with a distributive lattice the poset of its prime filters, ordered by reverse inclusion. It is possible to check that if D is a distributive lattice, then $\mathcal{O}(\mathrm{Spec}(D))$ is isomorphic to D, and if P is a poset, then $\mathrm{Spec}(\mathcal{O}(P))$ is isomorphic to P.

Example 5.3. Consider the distributive lattice L in Figure 5.3(1). The proper filters of L are $G_1 = \{b, c, d, e\}$, $G_2 = \{c, e\}$, $G_3 = \{d, e\}$, $G_4 = \{e\}$. One can check that G_1, G_2, G_3 are primes, while G_4 is not, because $c \vee d \in G_4$, but none of c, d is in G_4. Figure 5.3(2) shows the poset $\mathrm{Spec}(L)$, that is the poset of the prime filters G_1, G_2, G_3 ordered by reverse inclusion. Figure 5.3(3) shows the distributive lattice $\mathcal{O}(\mathrm{Spec}(L))$, that is the lattice of all downsets of $\mathrm{Spec}(L)$ ordered by inclusion. We immediately note that $\mathcal{O}(\mathrm{Spec}(L))$ is isomorphic to L.

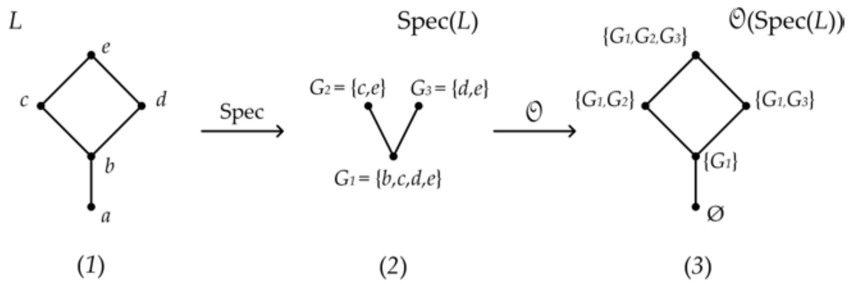


Figure 5.3: *Example 5.3.*

Let L, M be bounded distributive lattices, and $f : L \to M$ be a morphism of `DL`. We observe that, for each $G \in \mathrm{Spec}(M)$, $f^{-1}(G) \in \mathrm{Spec}(L)$. Then, we can define the dual map $f^* = \mathrm{Spec}(f) : \mathrm{Spec}(M) \to \mathrm{Spec}(L)$, by setting, for all $G \in \mathrm{Spec}(M)$

$$f^*(G) = f^{-1}(G). \tag{5.2}$$

One can verify that if $G, H \in \mathrm{Spec}(M)$ and $G \supseteq H$, then $f^{-1}(G) \supseteq f^{-1}(H)$. Thus f^* is an order-preserving map from $\mathrm{Spec}(M)$ to $\mathrm{Spec}(L)$. Furthermore, it is possible to prove that Spec is a functor from `DL` to `Poset`^{op}, and that it is an equivalence.

Alternatively to Spec, we can describe the equivalence between `DL` and `Poset`^{op} in a different manner, via the functor $\mathcal{J}$. To this end, we first define the *join irreducible elements* of a lattice L to be the elements $x \in L$, $x \neq \bot$, such that $y \vee z = x$ always imply $x = y$ or $x = z$.

For a distributive lattice L we define $\mathcal{J}(L)$ to be the poset of the join irreducible elements of L, ordered by the restriction of the order of L. The following important fact is easily verified.

Fact 5.2. *Let L be a distributive lattice. Then x is a join irreducible element of L if and only if $\uparrow x$ is a prime filter. Moreover, $\mathcal{J}(L)$ is isomorphic to* $\mathrm{Spec}(L)$ *via the map $\varphi : x \mapsto \uparrow x$.*

It turns out that any bounded distributive lattice L is isomorphic to $\mathcal{O}(\mathcal{J}(L))$ and, for any poset P, $P \cong \mathcal{J}(\mathcal{O}(P))$.

Consider now two bounded distributive lattices L and M, and let $\mathcal{J}(L) = P$ and $\mathcal{J}(M) = Q$. Let $f : L \to M$ be a $\mathtt{DL}$-morphism. We define $f^* = \mathcal{J}(f)$ to be the order-preserving map from Q to P defined by

$$f^*(y) = \min\{x \in P \mid y \in {\downarrow} f(x)\}, \tag{5.3}$$

for all $y \in Q$. We have so defined the action of the functor $\mathcal{J}$ both on objects and morphisms. It is not difficult to verify now that $\mathcal{J}$ is an equivalence from $\mathtt{DL}$ to $\mathtt{Poset}^{op}$.

By introducing the functors $\mathcal{O}$ and $\mathcal{J}$ (or, equivalently, Spec), we have described the dual equivalence between the categories $\mathtt{DL}$ and $\mathtt{Poset}$. This duality may be called *Birkhoff duality* for distributive lattices, cf. the bibliographic notes to this chapter.

The following example illustrates the action of the functor $\mathcal{J}$ on objects and morphisms. Compare 5.2.

Example 5.4. The top half of Figure 5.4 shows a morphism f between the bounded distributive lattices L and M. In the bottom half of the same figure we show the dual objects via $\mathcal{J}$, that is the posets $P = \mathcal{J}(L)$ and $Q = \mathcal{J}(M)$, and the dual morphism $f^* = \mathcal{J}(f)$, that is the order-preserving map $f^* : Q \to P$. Consider, for instance, the element $x \in Q$. By the definition of dual map – see (5.3) – $f^*(x) = \min\{y \in P \mid x \in {\downarrow} f(y)\}$. We note that ${\downarrow} f(a) = \{u, y\}$, ${\downarrow} f(b) = \{u, y, z\}$, ${\downarrow} f(c) = \{u, x, y, w\}$, ${\downarrow} f(d) = \{u, x, y, z, w, v\}$, and obtain $f^*(x) = c$.

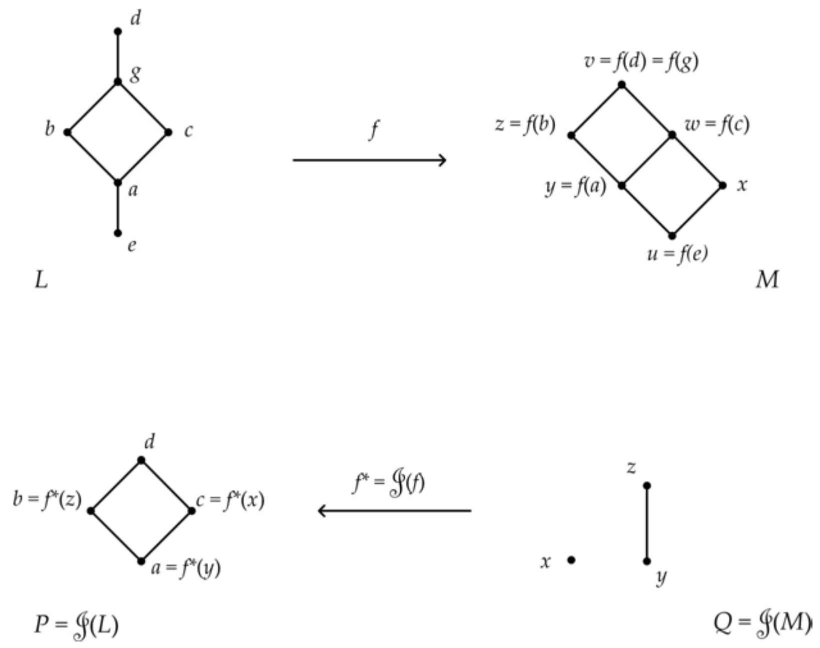


Figure 5.4: *Duality between* `Poset` *and* `DL` *via* $\mathcal{J}$.

The duality between `DL` and `Poset` suggest a new research direction. By investigating particular properties of the category `Poset`, we can now obtain dual results for the category `DL`. In particular, in the next sections we make use of the following important fact which allows us to translate the notions of monotone and regular partition into the corresponding notions for distributive lattices.

Fact 5.3. *Let P and Q be posets, and let $L = \mathcal{O}(P)$ and $M = \mathcal{O}(Q)$. Consider the order-preserving map $f : P \to Q$, and the map $f^* = \mathcal{O}(f)$ from M to L.*

1. *If f is epi, then f^* is mono.*
2. *If f is regular epi, then f^* is regular mono.*

A categorical proof of this fact – meaning a proof using arrows – is not difficult. We sketch the argument for the first case. Consider an epimorphism $P \xrightarrow{f} Q$ of `Poset`. By Definition 2.1, for all pairs $h, k : Q \to R$ of morphisms of `Poset` such that $h \circ f = k \circ f$, we have $h = k$ – see the diagram in Figure 5.5.

$$\begin{tikzcd} P \arrow[r, "f"] & Q \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & R \end{tikzcd}$$

Figure 5.5: *An epimorphism of* `Poset`.

We can translate this definition in terms of the dual category `DL`, using the fact that the functor $\mathcal{O}$ is in equivalence. We obtain that for all pairs $\mathcal{O}(h), \mathcal{O}(k) : \mathcal{O}(R) \to \mathcal{O}(Q)$ of morphisms of `DL` such that $\mathcal{O}(f) \circ \mathcal{O}(h) = \mathcal{O}(f) \circ \mathcal{O}(k)$, we have $\mathcal{O}(h) = \mathcal{O}(k)$ – see the diagram in Figure 5.6. Thus, by Definition 2.1, $\mathcal{O}(f)$ is a monomorphism of `DL`.

$$\begin{tikzcd} \mathcal{O}(P) & \mathcal{O}(Q) \arrow[l, "\mathcal{O}(f)"'] & \mathcal{O}(R) \arrow[l, shift right, "\mathcal{O}(h)"'] \arrow[l, shift left, "\mathcal{O}(k)"] \end{tikzcd}$$

Figure 5.6: *The dual of an epimorphism of* `Poset`.

To continue our work, we also need to reformulate the definition of $\{\mathbf{0}, \mathbf{1}\}$-sublattice used in the previous chapter.

Definition 5.9. Let L be an object of `DL`. We call S a $\{\mathbf{0}, \mathbf{1}\}$-*sublattice* of L if S is the range of a `DL`-monomorphism $f : M \to L$. If f is a regular monomorphism, we say that S is a *regular* $\{\mathbf{0}, \mathbf{1}\}$-*sublattice* of L.

It is well known that the set of all $\{\mathbf{0}, \mathbf{1}\}$-sublattices of a lattice L, indicated by $\mathrm{Sub}_{01}(L)$, forms a lattice.

5.3 Duals of monotone partitions

We indicate by $\Pi_m(P)$ the set of all monotone partitions of a poset P. As shown in Chapter 4, $\Pi_m(P)$ forms a lattice. The following result establishes a correspondence between the set $\Pi_m(P)$ and the set of all $\{\mathbf{0},\mathbf{1}\}$-sublattices of the lattice L that is dual to P via the functor $\mathcal{O}$.

Lemma 5.1. *There is a bijection between the set $\Pi_m(P)$ of all monotone partitions of a poset P and the set $\mathrm{Sub}_{01}(L)$ of all $\{\mathbf{0},\mathbf{1}\}$-sublattices of the distributive lattice $L = \mathcal{O}(P)$.*

Specifically, a bijection is obtained by associating with each $\pi \in \Pi_m(P)$, induced by a `Poset`*-epimorphism $P \xrightarrow{f} Q$, the $\{\mathbf{0},\mathbf{1}\}$-sublattice of $\mathcal{O}(P)$ which is the range of the* `DL`*-monomorphism $\mathcal{O}(Q) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(P)$.*

Proof. Let P be a poset, and let $(\pi, \preccurlyeq)$ be a monotone partition of P. Consider an epimorphism $f : P \to Q$ which induces the partition π according to Definition 3.3. Then, by Fact 5.3, $f^* = \mathcal{O}(f)$ is a monomorphism from $\mathcal{O}(Q)$ to $\mathcal{O}(P)$. By Definition 5.9, the range of f^* is a $\{\mathbf{0},\mathbf{1}\}$-sublattice of the distributive lattice $\mathcal{O}(P)$. We now observe that an epimorphism $g : P \to Q'$ yields the same monotone partition $(\pi, \preccurlyeq)$ induced by $f : P \to Q$ if and only if there exists an isomorphism $h : Q \to Q'$ which makes the diagram in Figure 5.7 commute. Thus, by duality, f and g induce the same monotone partition of P if and only if, for

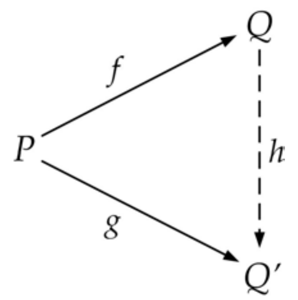


Figure 5.7: *Lemma 5.1 – monotone partition.*

$g^* = \mathcal{O}(g) : \mathcal{O}(Q') \to \mathcal{O}(P)$, there exists an isomorphism $h^* = \mathcal{O}(h) : \mathcal{O}(Q') \to \mathcal{O}(Q)$ which makes the diagram in Figure 5.8 commute, if and only if g^* yields the same $\{\mathbf{0},\mathbf{1}\}$-sublattice of $\mathcal{O}(P)$ induced by f^*. From this observation, and the fact that $\mathcal{O}$ is a dual equivalence, and hence faithful, full, and essentially surjective, we obtain our statement. □

We give an explicit example of the bijection provided by Lemma 5.1.

Example 5.5. Figure 5.9 shows an epimorphism f from a poset P to a poset Q. The partition $(\pi, \preccurlyeq)$ induced by f is depicted in the same figure, on the right. The dual of f via $\mathcal{O}$ is the monomorphism f^* between the bounded distributive lattices $\mathcal{O}(Q)$ and $\mathcal{O}(P)$. The monomorphism f^* induces a $\{\mathbf{0},\mathbf{1}\}$-sublattice $S \subseteq \mathcal{O}(P)$. Both f^* and S are depicted in Figure 5.10.

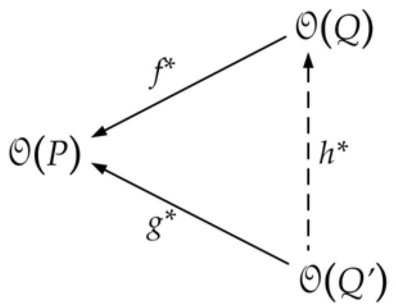


Figure 5.8: *Lemma 5.1 – $\{\mathbf{0},\mathbf{1}\}$-sublattice.*

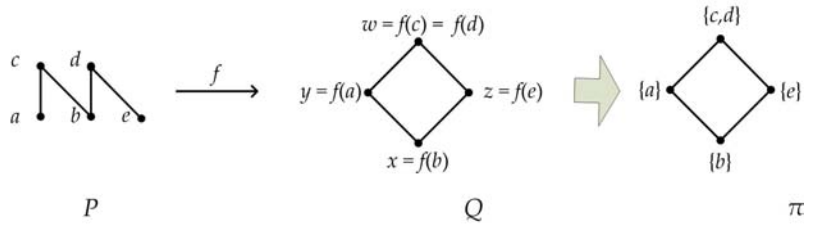


Figure 5.9: *A partition π induced by an epimorphism f.*

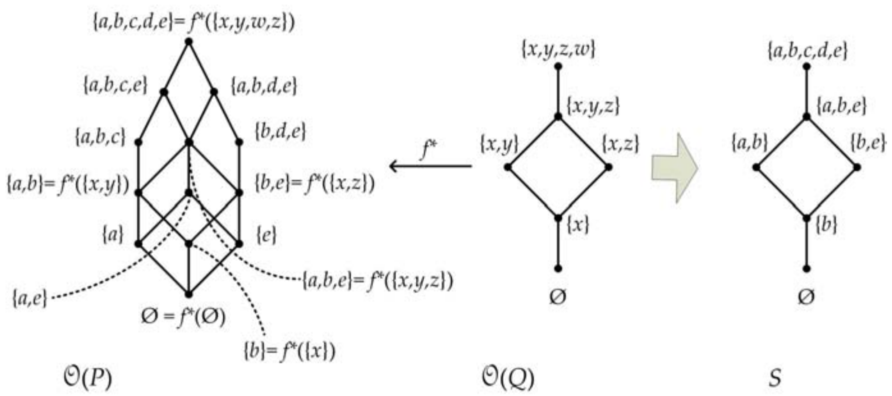


Figure 5.10: *A sublattice S which is range of a monomorphism f^*.*

We can define a lattice isomorphism φ which associates with the bounded distributive lattice $\mathcal{O}(\pi) = \{D_1, \dots, D_k\}$ the lattice $\varphi(\mathcal{O}(\pi)) = \{\overline{D_1}, \dots, \overline{D_k}\}$, where[1] $\overline{D_i} = \{x \in E \mid E \in D_i\}$, and $\overline{D_i} \leqslant \overline{D_j}$ in $\varphi(\mathcal{O}(\pi))$ if and only if $D_i \leqslant D_j$ in $\mathcal{O}(\pi)$, for all $D_i, D_j \in \mathcal{O}(\pi)$. We observe that $\varphi(\mathcal{O}(\pi)) = S$, as shown in Figure 5.11. One can verify that this fact holds in general, that is, for any poset P, $\varphi \circ \mathcal{O}$ is a bijection between $\Pi_m(P)$ and $\mathrm{Sub}_{01}(\mathcal{O}(P))$.

[1] Note that the elements of D_i are blocks of a partition, and thus sets.

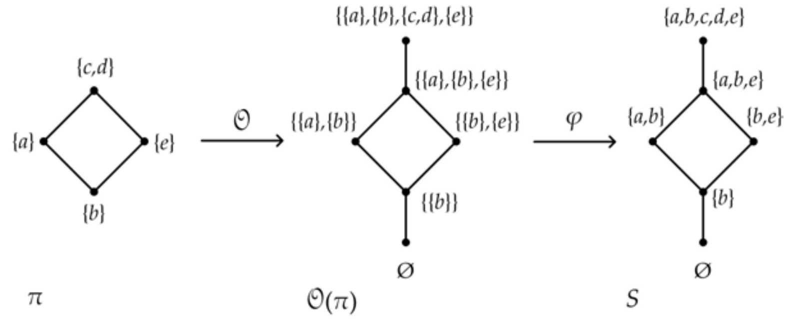


Figure 5.11: *A sublattice S dual to a partition π.*

In order to prove the main result of this section, one more lemma is needed.

Lemma 5.2. *Let P be a poset, and let $\pi,\pi' \in \Pi_m(P)$. Let $f : P \to Q$ and $g : P \to Q'$ be* `Poset`*-epimorphisms which induce π and π', respectively. Then $\pi \leqslant \pi'$ in $\Pi_m(P)$ if and only if there exists a* `Poset`*-epimorphism $e : Q \to Q'$ such that $e \circ f = g$.*

Proof. By Proposition 4.3, $\pi \leqslant \pi'$ if and only if the quasiorder $\lesssim$ corresponding to π is set-theoretically included in the quasiorder $\lesssim'$ corresponding to π'. Let us consider $\pi,\pi' \in \Pi_m(P)$ induced by the epimorphisms $f : P \to Q$ and $g : P \to Q'$, respectively.

Suppose $\pi \leqslant \pi'$. We define a map $e : Q \to Q'$ by setting for each $x \in Q$, $e(x) = g(f^{-1}(x))$. Since f is epi, for all $x \in Q$ we have $f^{-1}(x) \neq \emptyset$. Let $B = f^{-1}(x)$. If $p,q \in B$, then $p \lesssim q$ and $q \lesssim p$. Since $\lesssim \subseteq \lesssim'$ we also have $p \lesssim' q$ and $q \lesssim' p$ and thus $g(x) = g(y)$. Therefore e actually is a function from Q to Q'. Moreover, e is surjective, for else we would have for some $x \in Q'$, $e^{-1}(x) = \emptyset$, but there exists $p \in P$ such that $x = g(p) \neq e(f(p))$.

Let now $x,y \in Q$, with $x \leqslant y$. For all $p \in f^{-1}(x)$ and for all $q \in f^{-1}(y)$, we have $p \lesssim q$. Thus, we also have $p \lesssim' q$, and $g(p) = e(x) \leqslant g(q) = e(y)$. Therefore, e is order-preserving and thus an epimorphism.

Suppose now that there exists a `Poset`-epimorphism $e : Q \to Q'$ such that $e \circ f = g$, with the aim of proving $\pi \leqslant \pi'$. Let $p,q \in P$ be such that $p \lesssim q$. By Condition (4.1) in Proposition 4.1 we have $f(p) \leqslant f(q)$, and thus $e(f(p)) = g(p) \leqslant e(f(q)) = g(q)$. Using again Proposition 4.1 we obtain $p \lesssim' q$. Hence, $\lesssim \subseteq \lesssim'$ and, by Proposition 4.3, $\pi \leqslant \pi'$. □

For what sublattices are concerned, the order can be given in term of monomorphisms, as the following easy fact states.

Fact 5.4. *Let L be a bounded distributive lattice, and let $S,S' \in \mathrm{Sub}_{01}(L)$. Let $f : M \to L$ and $g : M' \to L$ be* `DL`*-monomorphisms having ranges S and S', respectively. Then $S' \leqslant S$ in $\mathrm{Sub}_{01}(L)$ if and only if there exists a* `DL`*-monomorphism $m : M' \to M$ such that $f \circ m = g$.*

Given two lattices L and M, we say that L is *anti-isomorphic* to M, written $L \overset{a}{\cong} M$, whenever the lattice L is isomorphic to the lattice obtained from M by reversing its order. The following theorem extends the result obtained in Lemma 5.1.

Theorem 5.1. *The monotone partition lattice of a poset P is anti-isomorphic to the lattice of all $\{\mathbf{0},\mathbf{1}\}$-sublattices of the distributive lattice $L = \mathcal{O}(P)$, ordered by inclusion. In symbols, $\Pi_m(P) \overset{a}{\cong} \mathrm{Sub}_{01}(L)$.*

Proof. Let P be a poset, and let π and π' be monotone partitions of P induced by the `Poset`-epimorphisms $f : P \to Q$ and $g : P \to Q'$, respectively. Let S and S' be the $\{\mathbf{0},\mathbf{1}\}$-sublattices of $L = \mathcal{O}(P)$ which are ranges of the `DL`-monomorphisms $f^* : \mathcal{O}(Q) \to \mathcal{O}(P)$ and $g^* : \mathcal{O}(Q') \to \mathcal{O}(P)$, respectively. By Lemma 5.2, $\pi \leqslant \pi'$ if and only if there exists a `Poset`-epimorphism $e : Q \to Q'$ such that $e \circ f = g$. By duality, $e \circ f = g$ if and only if $f^* \circ m = g^*$, with $m = \mathcal{O}(e)$, and, by Fact 5.4, this is equivalent to saying that S' is a $\{\mathbf{0},\mathbf{1}\}$-sublattice of S, that is $S' \leqslant S$. □

Example 5.6. Example 4.7 exhibits all the monotone partitions of a poset P, and their lattice structure. In Figure 5.12 we recall these results, showing the poset P together with its monotone partition lattice $\Pi_m(P)$. In Example 5.5 we have described the lattice $\mathcal{O}(P)$,

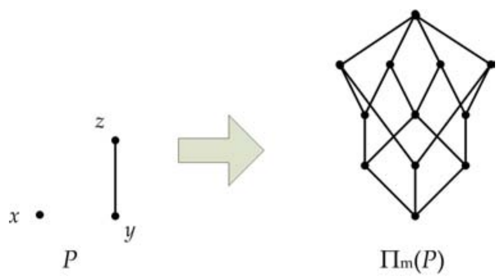


Figure 5.12: *Monotone partition lattice.*

dual to P. Figure 5.13 shows the sublattices of $\mathcal{O}(P)$. Finally, Figure 5.14 shows the lattice $\mathrm{Sub}_{01}(\mathcal{O}(P)) \overset{a}{\cong} \Pi_m(P)$.

A very extensive body of literature is concerned with Birkhoff duality and its extensions. A part of this literature deals with the lattice of sublattices of distributive lattices, both for the finite and infinite case. We give a brief description of these latter results in the bibliographic notes, at the end of this chapter.

5.4 Duals of regular partitions

The results obtained in the previous section for monotone partition lattices can be adapted in order to obtain analogous results for regular partition lattices. We do not give proofs in this section, because they are straightforward translations of the proofs given for the corresponding results in Section 5.3. Essentially, it suffices to substitute the term *monotone* with the term *regular*, and the notion of $\{\mathbf{0},\mathbf{1}\}$-*sublattice* with that of *regular* $\{\mathbf{0},\mathbf{1}\}$-*sublattice*. Given a poset P, we indicate by $\Pi_r(P)$ the regular partition lattice of P. We indicate by $\mathrm{Sub}_r(L)$ the set of all regular $\{\mathbf{0},\mathbf{1}\}$-sublattices of a bounded distributive lattice L.

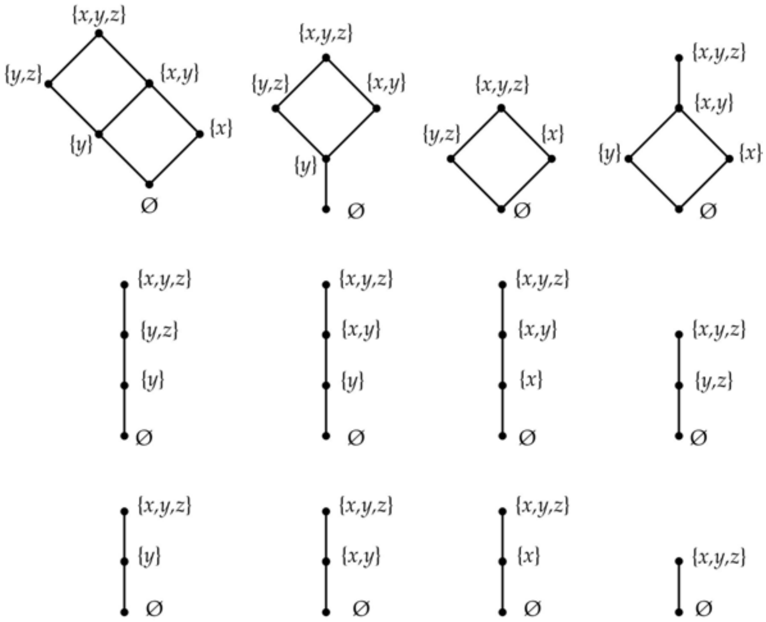


Figure 5.13: $\{\mathbf{0},\mathbf{1}\}$-*sublattices of a bounded distributive lattice.*

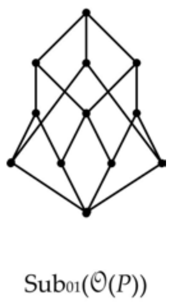


Figure 5.14: *Lattice of sublattices of a bounded distributive lattice.*

Lemma 5.3. *There is a bijection between the set* $\Pi_r(P)$ *of all regular partitions of a poset* P *and the set* $\mathrm{Sub}_r(L)$ *of all regular* $\{\mathbf{0},\mathbf{1}\}$*-sublattices of the distributive lattice* $L = \mathcal{O}(P)$*.*

Specifically, a bijection is obtained by associating with each $\pi \in \Pi_r(P)$*, induced by a regular* `Poset`*-epimorphism* $P \xrightarrow{f} Q$*, the regular* $\{\mathbf{0},\mathbf{1}\}$*-sublattice of* $\mathcal{O}(P)$ *which is the range of the regular* `DL`*-monomorphism* $\mathcal{O}(Q) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(P)$*.*

The following lemma expresses the order between regular partitions in terms of the corresponding regular epimorphisms.

Lemma 5.4. *Let P be a poset, and let $\pi, \pi' \in \Pi_r(P)$. Let $f : P \to Q$ and $g : P \to Q'$ be regular* `Poset`*-epimorphisms which induce π and π', respectively. Then $\pi \leqslant \pi'$ in $\Pi_r(P)$ if and only if there exists a regular* `Poset`*-epimorphism $e : Q \to Q'$ such that $e \circ f = g$.*

For what concern regular sublattices the order can be given in terms of regular monomorphisms, as follows.

Fact 5.5. *Let L be a bounded distributive lattice, and let $S, S' \in \mathrm{Sub}_r(L)$. Let $f : M \to L$ and $g : M' \to L$ be regular* `DL`*-monomorphisms having ranges S and S', respectively. Then $S' \leqslant S$ in $\mathrm{Sub}_r(L)$ if and only if there exists a regular* `DL`*-monomorphism $m : M' \to M$ such that $f \circ m = g$.*

The following theorem establishes the desired correspondence between the regular partition lattice of a poset P and the lattice of regular $\{\mathbf{0}, \mathbf{1}\}$-sublattices of the distributive lattice dual to P.

Theorem 5.2. *The regular partition lattice of a poset P is anti-isomorphic to the lattice of all regular $\{\mathbf{0}, \mathbf{1}\}$-sublattices of the distributive lattice $L = \mathcal{O}(P)$, ordered by inclusion. In symbols, $\Pi_r(P) \overset{a}{\cong} \mathrm{Sub}_r(L)$.*

Example 5.7. Consider the poset P and the distributive lattice $\mathcal{O}(P)$ depicted in Figure 5.15. The regular sublattices of $\mathcal{O}(P)$, shown in Figure 5.16, are the duals of the regular partitions

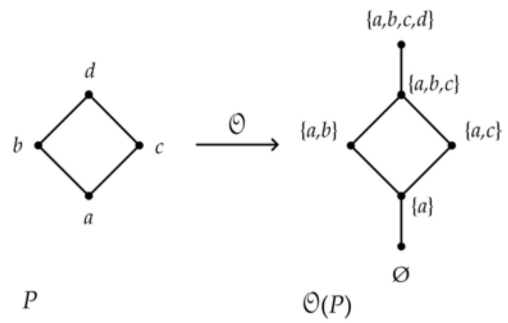


Figure 5.15: *A poset P and its dual $\mathcal{O}(P)$.*

of P shown in Figure 4.13 – see the construction of the dual of a partition in Example 5.5. Comparing the lattice $\mathrm{Sub}_r(\mathcal{O}(P))$ depicted in Figure 5.17 with the lattice $\Pi_r(P)$ shown in Figure 4.14, we see that $\Pi_r(P) \overset{a}{\cong} \mathrm{Sub}_r(\mathcal{O}(P))$.

5.5 Characterization of regular sublattices

The aim of this section is to give a characterization of regular sublattices. To this end, we will translate the characterization of a regular partition of a poset, given in Chapter 3, in the

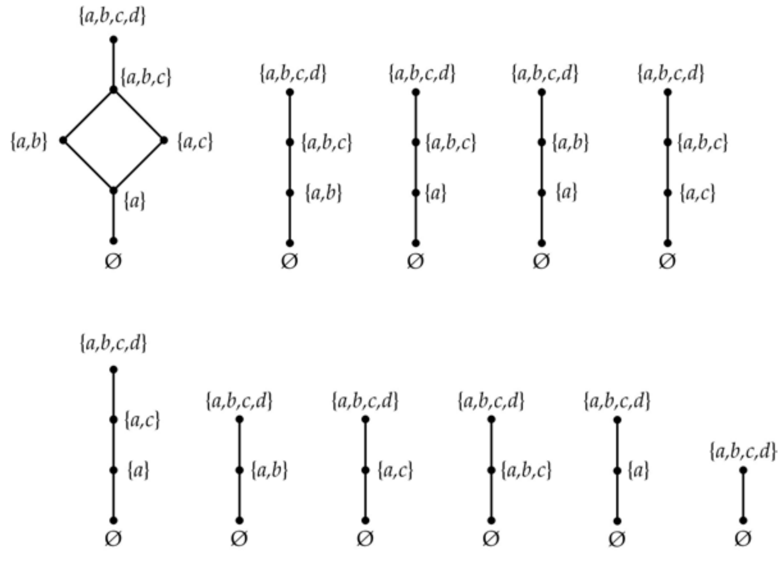
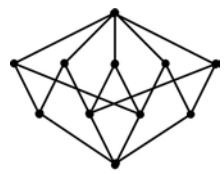


Figure 5.16: *Regular* $\{\mathbf{0},\mathbf{1}\}$*-sublattices of a bounded distributive lattice.*



$\mathrm{Sub}_r(\mathcal{O}(P))$

Figure 5.17: *Lattice of regular sublattices of a bounded distributive lattice.*

dual category DL. We first show how to relate join-irreducible elements of a sublattice with blocks of the dual partition.

Let P be a poset, and let $(\pi, \preccurlyeq)$ be a monotone partition of P. Consider the distributive lattice $L = \mathcal{O}(P)$, and the $\{\mathbf{0},\mathbf{1}\}$-sublattice $M \subseteq L$ dual to π, that is, the sublattice of L associated with π by $\mathcal{O}$ according to Lemma 5.1. By the definition of the functor $\mathcal{J}$, we can think of the poset P as the poset of join-irreducible elements of L. Consequently, the set partition $\pi = \{B_1, \ldots, B_k\}$ can be easily associated with a partition of the set $\mathcal{J}(L)$. The next definition, as will be shown in Lemma 5.5 below, describes the blocks of such a partition.

Notation. If L is a bounded distributive lattice, and $M \subseteq L$ a $\{\mathbf{0},\mathbf{1}\}$-sublattice of L, we denote by π_M the dual of the sublattice M, that is, the monotone partition of the poset $\mathcal{J}(L)$ corresponding to the sublattice M according to Lemma 5.1.

Definition 5.10. Let L be a bounded distributive lattice, and $M \subseteq L$ a sublattice of L. Fix

$m \in \mathfrak{J}(M)$. We call the *block of m* the subset $B_m \subseteq \mathfrak{J}(L)$ defined by

$$B_m = (\downarrow m \cap \mathfrak{J}(L)) \setminus \downarrow ((\mathfrak{J}(M) \cap \downarrow m) \setminus \{m\}), \tag{5.4}$$

where $\downarrow$ denotes downsets in the lattice L.

The next example should clarify this definition.

Example 5.8. Consider the distributive lattice L and the $\{\mathbf{0},\mathbf{1}\}$-sublattice $M \subseteq L$ depicted in Figure 5.18.

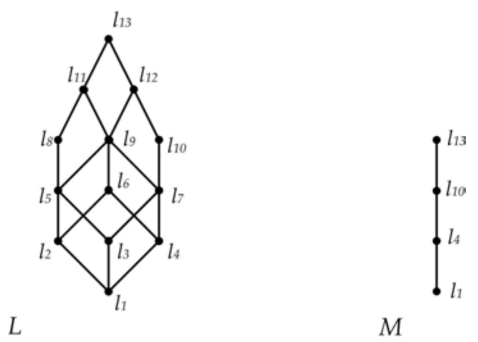


Figure 5.18: *Example 5.8.*

We observe that $\mathfrak{J}(L) = \{l_2, l_3, l_4, l_8, l_{10}\}$ and $\mathfrak{J}(M) = \{l_4, l_{10}, l_{13}\}$. Consider the element $l_{10} \in \mathfrak{J}(M)$. By Definition 5.10 we have

$$B_{l_{10}} = (\downarrow l_{10} \cap \{l_2, l_3, l_4, l_8, l_{10}\}) \setminus \downarrow ((\{l_4, l_{10}, l_{13}\} \cap \downarrow l_{10}) \setminus \{l_{10}\}) =$$
$$= (\{l_1, l_3, l_4, l_7, l_{10}\} \cap \{l_2, l_3, l_4, l_8, l_{10}\}) \setminus \downarrow ((\{l_4, l_{10}, l_{13}\} \cap \{l_1, l_3, l_4, l_7, l_{10}\}) \setminus \{l_{10}\}) =$$
$$= \{l_3, l_4, l_{10}\} \setminus \downarrow (\{l_4, l_{10}\} \setminus \{l_{10}\}) = \{l_3, l_4, l_{10}\} \setminus \downarrow l_4 = \{l_3, l_4, l_{10}\} \setminus \{l_1, l_4\} = \{l_3, l_{10}\}.$$

In the same way we can obtain $B_{l_4} = \{l_4\}$, and $B_{l_{13}} = \{l_2, l_8\}$. Note that the set $\{B_{l_4}, B_{l_{10}}, B_{l_{13}}\}$ is a partition of $\mathfrak{J}(L)$.

Lemma 5.5. *Let L be a bounded distributive lattice, let $M \subseteq L$ be a $\{\mathbf{0},\mathbf{1}\}$-sublattice of L, and let $\mathfrak{J}(M) = \{m_1, \ldots, m_k\}$. Then*

$$\pi_M = \{B_{m_1}, \ldots, B_{m_k}\},$$

where B_{m_i} is the block of m_i.

Proof. Let $f : M \to L$ be the canonical embedding of M into L, that is the map from M to L such that for all $x \in M$, $f(x) = x$. By the definition of $\mathfrak{J}$ the dual of f is the map $f^* : \mathfrak{J}(L) \to \mathfrak{J}(M)$ defined by

$$f^*(y) = \min\{x \in \mathfrak{J}(M) \mid y \in \downarrow f(x)\},$$

for all $y \in \mathfrak{J}(L)$ – see (5.3). The map f^* is an order-preserving surjection from $\mathfrak{J}(L)$ to $\mathfrak{J}(M)$. We claim that the fibres of f^* are exactly $B_{m_1}, \ldots, B_{m_k}$. To be more precise, we claim that for all $m \in \mathcal{J}(M)$, $B_m = (f^*)^{-1}(m)$. Consider an element $z \in \mathcal{J}(L)$. We observe that $z \in (f^*)^{-1}(m)$ if and only if $m = \min\{x \in \mathcal{J}(M) \mid z \in \downarrow f(x)\} = \min\{x \in \mathcal{J}(M) \mid z \in \downarrow x\}$. Thus, $(f^*)^{-1}(m)$ is the set of all elements $z \in \mathcal{J}(L) \cap \downarrow m$ such that there is no $m' \in \mathcal{J}(M)$, with $m' < m$, such that $z \in \downarrow (m' \cap \mathcal{J}(L))$. Noting that the set of all $m' \in \mathcal{J}(M)$ such that $m' < m$ is the set $(\mathcal{J}(M) \cap \downarrow m) \setminus \{m\}$, we can write $(f^*)^{-1}(m) = (\mathcal{J}(L) \cap \downarrow m) \setminus (\downarrow ((\mathcal{J}(M) \cap \downarrow m) \setminus \{m\}) \cap \mathcal{J}(L)) = (\mathcal{J}(L) \cap \downarrow m) \setminus \downarrow ((\mathcal{J}(M) \cap \downarrow m) \setminus \{m\}) = B_m$. By Lemma 5.1, $\pi_M = \{(f^*)^{-1}(m) \mid m \in \mathcal{J}(M)\} = \{B_{m_1}, \ldots, B_{m_k}\}$, as desired. □

Consider now a distributive lattice L and a $\{\mathbf{0},\mathbf{1}\}$-sublattice $M \subseteq L$. Lemma 5.5 says that if $\mathcal{J}(M) = \{m_1, \ldots, m_k\}$, then $\{B_{m_1}, \ldots, B_{m_k}\}$ is the underlying set of a monotone partition of the poset $\mathcal{J}(L)$. Since each join-irreducible element of the sublattice M represents a block of this partition, the order between blocks can be recovered by looking at the order of the corresponding join-irreducible elements of M. By such considerations, it is not difficult to translate the condition in Theorem 3.2, where a characterization of regular partitions is given, into a corresponding condition on the lattice L. The following corollary provides a first characterization of regular sublattices along these lines.

Corollary 5.3. *Let L be a bounded distributive lattice, let $M \subseteq L$ be a $\{\mathbf{0},\mathbf{1}\}$-sublattice of L, and let $\mathcal{J}(M) = \{m_1, \ldots, m_k\}$. Consider the poset $(\pi, \preccurlyeq)$, where $\pi = \{B_{m_1}, \ldots, B_{m_k}\}$, and where $\preccurlyeq$ is a partial order on π defined by*

$$m_i \leqslant m_j \text{ if and only if } B_{m_i} \preccurlyeq B_{m_j}, \tag{5.5}$$

for all $i, j \in \{1, \ldots, k\}$. Then, M is a regular $\{\mathbf{0},\mathbf{1}\}$-sublattice of L if and only if for all $i, j \in \{1, \ldots, k\}$ and for all $x \in B_{m_i}$, $y \in B_{m_j}$ the following condition holds.

$$B_{m_i} \preccurlyeq B_{m_j} \text{ if and only if } x \lesssim_\pi y, \tag{5.6}$$

where $\lesssim_\pi$ is the blockwise quasiorder induced by π on $\mathcal{J}(L)$.

Proof. By Lemma 5.5, $\pi = \{B_{m_1}, \ldots, B_{m_k}\}$ is the monotone partition of $\mathcal{J}(L)$ dual to M. By duality (Fact 5.3 and Definition 5.9), M is a regular sublattice of L if and only if π is a regular partition of $\mathcal{J}(L)$. Thus, by Theorem 3.2, π is a regular sublattice of L if and only Condition (5.6) holds. □

Example 5.9. Consider the lattice L and its sublattice $M \subseteq L$, depicted in Figure 5.18. Example 5.8 shows that M induces on $\mathcal{J}(L)$ the partition $\pi = \{B_{l_4}, B_{l_{10}}, B_{l_{13}}\}$, where $B_{l_4} = \{l_4\}$, $B_{l_{10}} = \{l_3, l_{10}\}$, and $B_{l_{13}} = \{l_2, l_8\}$. Using Condition (5.5) in Corollary 5.3 we define an order on π, obtaining

$$B_{l_4} \preccurlyeq B_{l_{10}} \preccurlyeq B_{l_{13}}.$$

Since $l_4 \lesssim_\pi l_3 \lesssim_\pi l_2$, Condition (5.6) in Corollary 5.3 is satisfied. Thus, M is a regular $\{\mathbf{0},\mathbf{1}\}$-sublattice of L.

Consider now the lattice L' and its sublattice M', depicted in Figure 5.19. We have $\mathcal{J}(L') = \{b, c, d\}$ and $\mathcal{J}(M') = \{c, f\}$. By Lemma 5.5, M' induces on $\mathcal{J}(L')$ the partition $\pi' =$

$\{B_c, B_f\}$, where $B_c = \{c\}$ and $B_f = \{b,d\}$. Using Condition (5.5) in Corollary 5.3 we define on π' a partial order, obtaining $B_c \preccurlyeq B_f$, because $c \leqslant f$. Since $B_c \preccurlyeq B_f$, but $c \nleqslant_{\pi'} b$, by Corollary 5.3 M' is not a regular sublattice of L'.

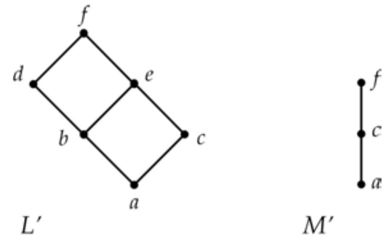


Figure 5.19: *Example 5.9.*

Corollary 5.3 gives a first characterization of the regular partition lattices. A more algebraic characterization is given by the following theorem.

Theorem 5.4. *Let L be a bounded distributive lattice, and let $M \subseteq L$ be a $\{\mathbf{0},\mathbf{1}\}$-sublattice of L. Let $[M]$ be the class of all sublattices $M' \subseteq L$ such that $\pi_M = \pi_{M'}$. Then M is a regular sublattice of L if and only if*

$$\bigvee [M] = M\,,$$

where the join is computed in $\mathrm{Sub}_{01}(L)$ *– that is, M is the sublattice of L generated by* $[M]$.

Proof. Consider the monotone partitions $(\pi_M, \preccurlyeq)$ and $(\pi_{M'}, \preccurlyeq')$ of $\mathcal{J}(L)$, and suppose $\pi_M = \pi_{M'}$. Recall that, by Proposition 4.3, the lattice $\Pi_m(\mathcal{J}(L))$ is ordered by set-theoretical inclusion between the quasiorders corresponding to each monotone partition. By Theorems 3.1 and 3.2, we infer that if π_M is regular, then $\pi_M \leqslant \pi_{M'}$. Thus, if we denote by $[\pi_M]$ the class of all monotone partitions having the same underlying set as π_M, we have that

$$\pi_M \text{ is a regular partition if and only if } \pi_M = \bigwedge [\pi_M]\,,$$

where the meet is computed in $\Pi_m(\mathcal{J}(L))$. By Theorem 5.1, the lattice $\mathrm{Sub}_{01}(L)$ is anti-isomorphic to $\Pi_m(\mathcal{J}(L))$. Then, it follows that M is a regular sublattice of L if and only if $\bigvee[M] = M$, where the join is computed in $\mathrm{Sub}_{01}(L)$. □

This is not a conclusive result for what concerns the characterization of regular sublattices. An intrinsic algebraic characterization of this class which does not mention the construction of the dual partition should be feasible. Such a characterization is left for further work.

5.6 Bibliographic notes

The notions concerning categories used in this chapter can be found in almost every book on category theory. We cite, for instance, [HS73], [AHS04], [AM75], [Man76], [ML98].

As a general reference for the duality between distributive lattices and posets (with extensions to the infinite case) we mention [DP02] and [Pri84]. In this chapter we have referred to the finite case of this duality as *Birkhoff duality*, after [Bir40]. The general case is sometimes called, depending on the context, *Priestley duality*, *Stone duality*, or *Stone-Priestley duality*.

The literature concerned with this duality is vast. Here, we only mention some papers that are more closely related to our work.

In [CLP91], R. Cignoli, S. Lafalce and A. Petrovich present a systematic account of relations on compact totally order-disconnected spaces (*Priestley spaces*) and a variety of applications. They set up a duality for bounded distributive lattices with $\mathbf{0}$-preserving and join-preserving maps. They also establish a duality between $\{\mathbf{0}, \mathbf{1}\}$-sublattices of a bounded distributive lattice L and suitable preorder relations of the Priestley space of L. This can be considered as a more general framework within which the present thesis fits.

In [ADS96], the authors use Birkhoff duality to investigate maximal sublattices of finite bounded distributive lattices. They use *quotients* instead of *monotone partitions* and study the set of all atoms of the lattice $Q(P)$ of quotients. Via Birkhoff duality, $Q(P)$ is dually isomorphic to the lattice of sublattices of $L = \mathcal{O}(P)$. The set of all atoms of $Q(P)$ thus corresponds to the set of maximal proper sublattices of L. Amongs other results, some arithmetical properties concerning the number and size of maximal proper sublattices of a finite distributive lattice are obtained.

In [Sch99] and [Sch02] J. Schmid continues the study of the lattices of all sublattices of a finite bounded distributive lattice. In the first paper the author discusses, with the aid of Birkhoff duality, the *remainder* $R = L \setminus M$, of the finite distributive lattice L, where M is a maximal sublattice of L. In [Sch02] a new tool for the investigation of $\{\mathbf{0}, \mathbf{1}\}$-sublattices of a bounded distributive lattice is presented. This idea replaces a partial order in the definition of a Priestley space with a compatible quasiorder. As a consequence new proofs of representations of different sublattices are derived. The author investigates special sublattices, such as *epic*, *Frattini* and maximal sublattices.

In [Ada73] M. E. Adams considers the Frattini sublattice $\Phi(L)$ of a lattice L and proves, among other things, that if L is a distributive lattice then there exists a distributive lattice L_1 such that L is isomorphic to $\Phi(L_1)$. Again, the methods are based on Priestley duality. In [AA94] distributive lattices which are the Frattini sublattices of finite distributive lattices are characterized by means of a property of the ordered set of all join-irreducible elements.

The lattice of all sublattices of a distributive lattice is also investegated, for instance, in [Riv73] and [Riv74]. Its dual, that is the monotone partition lattice of a poset (or, equivalently, the lattice of quasiorders on a set, or a particular lattice of topologies) are studied, for instance, in [ER95], [ER96], [Ric98].

Chapter 6

Counting problems

Si sta come, d'autunno, sugli alberi, le foglie.

Giuseppe Ungaretti

6.1 Monotone and regular partitions of antichains

This last chapter is devoted to some counting problems arising from the theory developed in this thesis. We do not tackle general enumerative problems, but simply introduce some cases to suggest that this field of research might be vast and possibly difficult. References on this topic are given in the bibliographic notes at the end of the chapter.

We begin with the simplest case of all, that is the enumeration of monotone and regular partitions of antichains. We consider an antichain A_n with $n \geqslant 0$ elements, with the aim of counting the number of its monotone and regular partitions.

Any quasiorder on the elements of A_n must extend the partial order of A_n, because the latter is just the identity relation on A_n. Therefore, by Proposition 4.1, the number of monotone partitions of A_n equals the number of quasiorders on n elements. We indicate this number by Q_n. There is no known explicit formula for Q_n. In the following table we display the value of Q_n, for $n \leqslant 8$.

n	0	1	2	3	4	5	6	7	8
Q_n	1	1	4	29	355	6942	209527	9535241	642779354

We note that the collection of all quasiorders on n elements can be obtained by partitioning the n elements in all possible ways, and then defining in all possible ways a partial order between the blocks of each partition. We can thus derive the following formula for Q_n, which, clearly, does not simplify the problem of computing its values:

$$Q_n = \sum_{k=1}^{n} S(n,k) P_k \,.$$

Here, $S(n,k)$ counts the number of partitions of a set of n elements into k blocks, and is called the *Stirling number of the second kind*, while P_k is the number of partial orders on n elements. As for Q_n, there seems to be no explicit formulas to compute P_n. The values of P_n for $n \leqslant 8$ are given by the following table.

n	0	1	2	3	4	5	6	7	8
P_n	1	1	3	19	219	4231	130023	6129859	431723379

Example 6.1. Figure 6.1 shows all monotone partitions of the antichain $A_2 = \{x, y\}$.

Figure 6.1: *Monotone partitions of* A_2.

Figure 6.2 shows the lattice of monotone partitions of A_2, that is, the lattice of quasiorders on 2 elements.

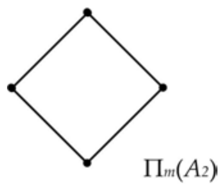


Figure 6.2: *Monotone partition lattice of* A_2.

To compute the number of regular partitions of an antichain with n points, we use Corollary 3.4. We immediately observe that the condition in this corollary is satisfied by every (set) partition of an antichain, that is, every partition of an antichain A_n admits an extension to a regular partition of A_n. Since this extension is unique, the number of regular partitions of A_n equals the number of partitions of a set with n elements. This number is known as the n-th Bell number, and denoted by B_n. By the definition of Stirling numbers of the second kind, we have

$$B_n = \sum_{k=0}^{n} S(n,k).$$

The following table shows the values of B_n for $n \leqslant 10$.

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	1	2	5	15	52	203	877	4140	21147	115975

Remark 6.1. By the latter considerations, it is clear that $\Pi_r(A_n)$ is isomorphic to the partition lattice Π_n.

Example 6.2. Figure 6.3 shows the regular partition lattices of antichains with $1 \leqslant n \leqslant 4$ points, that is the lattices of partitions of a set with n elements.

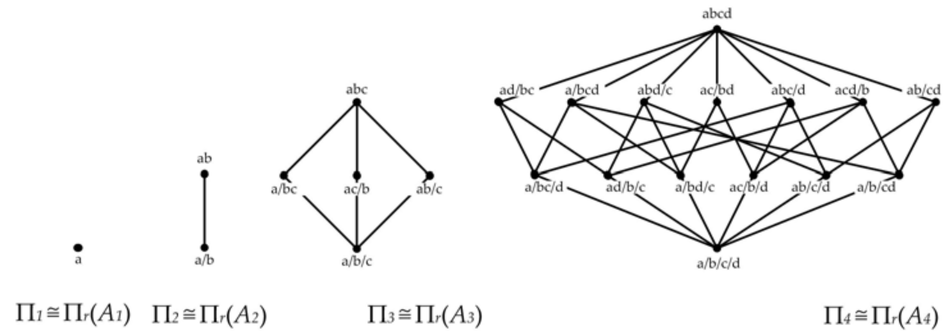


Figure 6.3: *Regular partitions of antichains.*

6.2 Monotone and regular partitions of chains

The second case we deal with is the enumeration of monotone and regular partitions of chains. We indicate by C_n a chain with n elements.

Proposition 6.1. *Let π be a (set) partition of a chain C_n. Then π admits an extension to a regular partition of C_n if and only if each block $B \in \pi$ is an interval,* i.e. *there exist $x, y \in C_n$ such that*

$$B = [x, y] = \{z \in C_n \mid x \leqslant z \leqslant y\}.$$

Proof. Let π be a partition of a chain C_n.

($\Leftarrow$) Suppose π is a partition of C_n into intervals. One can immediately see that $x \lesssim_\pi y$ and $y \lesssim_\pi x$ if and only if x and y belong to the same block of π. By Corollary 3.4, π admits an extension to a regular partition of C_n.

($\Rightarrow$) Suppose now that there exists a block $B \in \pi$ which is not an interval. Thus, there exist $x \leqslant z \leqslant y$, with $x, y \in B$ and $z \notin B$. We thus obtain $x \lesssim_\pi z \lesssim_\pi y \lesssim_\pi x$, with x and z not belonging to the same block. By Corollary 3.4, π does not admit an extension to a regular partition of C_n. □

By Corollary 3.4, if a partition π admits an extension to a regular partition, this extension is unique. Thus, by Proposition 6.1, enumerating regular partitions of a chain C_n is equivalent to enumerating partitions of C_n into intervals. It is possible to establish a bijection between such partitions and the *compositions*, also called *ordered partitions*, of n, that is the expressions of n as *ordered* sums of positive integers. It is easy to check the well-known fact that the number of ordered partitions of n is 2^{n-1}. Thus, the number of regular partitions of C_n also equals the number of subsets of a set with $n-1$ elements. In fact, one can identify a partition of C_n into intervals with a subset of edges (*i.e.* pairs (x,y) of elements such that y covers x) of C_n, as shown in the following example.

Example 6.3. Consider the chain C_4 – see Figure 6.4 – and choose a subset of its edges, $S = \{(a,b),(b,c)\}$, say. Then, S defines a partition $\pi = \{B_1, B_2, B_3\}$, where $B_1 = \{x \in C_n \mid x \leqslant a\}$,

$B_2 = \{x \in C_n \mid b \leqslant x \leqslant b\}$, $B_3 = \{x \in C_n \mid c \leqslant x\}$. As shown in Figure 6.4, π is a partition of C_n into intervals.

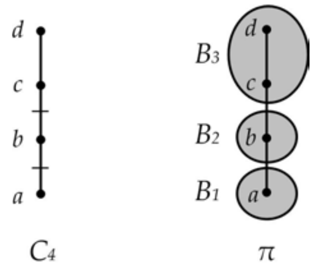


Figure 6.4: *Regular partition of a chain.*

Remark 6.2. By the latter considerations, we obtain that the lattice $\Pi_r(C_n)$ is isomorphic to the lattice $\mathcal{B}_{n-1}$ of subsets of a set with $n-1$ elements, ordered by inclusion, called a *Boolean lattice*.

Example 6.4. Figure 6.5 shows the lattices $\Pi_r(C_n) \cong \mathcal{B}_{n-1}$ for $n = 1,2,3,4$.

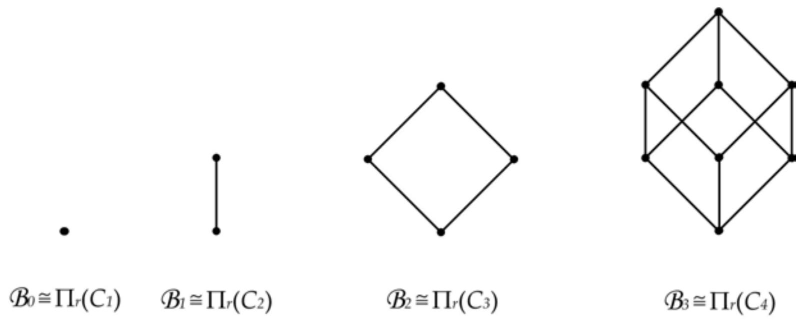


Figure 6.5: *Regular partition lattices of chains.*

Consider now a chain C_n and a regular partition $(\pi = \{B_1, \dots, B_k\}, \leqslant)$ of C_n. By Theorem 3.2, if $B_i \leqslant B_j$, for $i, j \in \{1, \dots, k\}$, then $x \lesssim_\pi y$, for any $x \in B_i$ and $y \in B_j$. Since π is a partition of C_n into intervals, $x \lesssim_\pi y$ if and only if $x \leqslant y$ or $B_i = B_j$. Thus, for a chain C_n, Condition (3.2) in Theorem 3.2 coincide with Condition (3.1) in Theorem 3.1. We obtain that, for chains, monotone partitions and regular partitions coincide:

$$\Pi_m(C_n) = \Pi_r(C_n)\,.$$

From the results obtained in this section and in the previous one, we can easily derive the following proposition, which provides bounds for the number of regular and monotone partitions of any poset.

Proposition 6.2. *Let P be a poset with n elements. Then*

$$2^{n-1} \leqslant |\Pi_r(P)| \leqslant B_n\,,$$

$$2^{n-1} \leqslant |\Pi_m(P)| \leqslant Q_n\,.$$

Proof. The inequalities for the monotone partition lattice are immediately obtained, by Proposition 4.1, using quasiorders. The inequality $|\Pi_r(P)| \leqslant B_n$ follows directly from Corollary 3.4. Consider now a poset P and a partition π of its underlying set. Extend the order of P to a total order, obtaining a chain C_n. Thus, $x \lesssim_\pi y$ in P always implies $x \lesssim_\pi y$ in C_n. By Corollary 3.4, whenever π admits an extension to a regular partition of C_n, it admits an extension to a regular partition of P. Since $|\Pi_r(C_n)| = 2^{n-1}$, the remaining inequality is proved. □

6.3 Monotone and regular partitions of linear sums

In this section we provide a method to count monotone and regular partitions of some posets which are obtained through *linear sums*.

Definition 6.1. Let P and Q be posets. The *linear sum* $P \oplus Q$ is the poset $P \cup Q$, with the order defined by

$x \leqslant y$ if and only if $x, y \in P$ and $x \leqslant y$ in P, or $x, y \in Q$ and $x \leqslant y$ in Q, or $x \in P, y \in Q$.

The Hasse diagram of a poset $P \oplus Q$ is obtained by placing the diagram of P directly below the diagram of Q, and then by adding a line segment from each maximal element of P to each minimal element of Q, as shown in Figure 6.6. Observe that, while the linear sum is not commutative, it is associative.

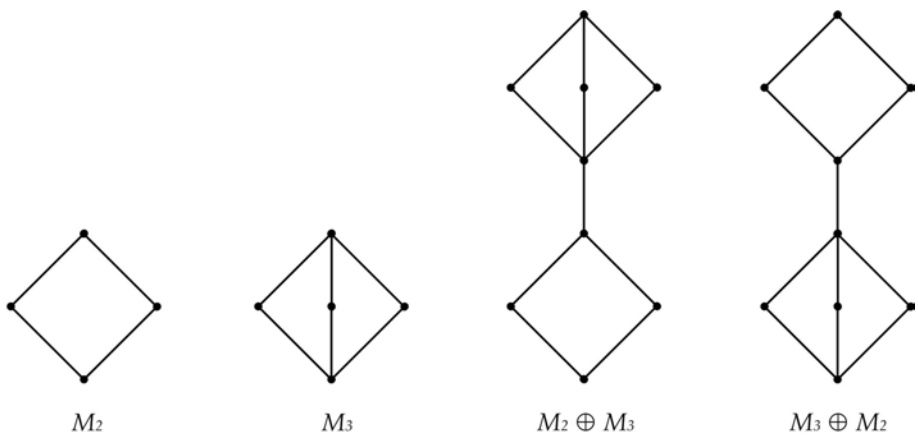


Figure 6.6: *Linear sum of posets.*

Example 6.5. An interesting example of posets obtainable by linear sum is the family of posets $T_i = A_1 \oplus A_i$, that we call *bushes*. Note that bushes are just trees of height 1. In analogy with our analysis of antichains in Section 6.1, we observe that all partitions of the underlying set of a bush T_i admit an extension to a regular partition of T_i. Since T_i has $i+1$ elements, we have, for $i = 1, 2, \ldots$,

$$\Pi_r(T_i) \cong \Pi_r(A_{i+1}) \cong \Pi_{i+1}\,.$$

We recall a well-known notion.

Definition 6.2. Let P and Q be posets. The *cartesian product* of P and Q is the poset

$$P \times Q = \{(x,y) | x \in P, y \in Q\}\,.$$

The order on $P \times Q$ is defined by

$$(x,y) \leqslant (w,z) \text{ if and only if } x \leqslant y \text{ in } P \text{ and } y \leqslant z \text{ in } Q\,.$$

The following proposition provides the promised method to count monotone and regular partitions of linear sums of some special posets. This method also applies to the chains studied in Section 6.2 in an obvious manner.

Proposition 6.3. *Let P be a poset with top and Q be a poset with bottom. Then*

$$\Pi_r(P \oplus Q) \cong \Pi_r(P) \times \Pi_r(Q) \times C_2\,,$$

$$\Pi_m(P \oplus Q) \cong \Pi_m(P) \times \Pi_m(Q) \times C_2\,.$$

Proof. Let P, Q be posets, and let b be the bottom of Q and t the top of P. Let π be a regular partition of $P \oplus Q$. Let $x \in P$ and $y \in Q$, and suppose that there exists a block $B \in \pi$ such that $x, y \in B$. By Corollary 3.4, we have $x \lesssim_\pi y$ and $y \lesssim_\pi x$. We also have $x \lesssim_\pi t \lesssim_\pi b \lesssim_\pi y \lesssim_\pi x \lesssim_\pi t$. Thus, by Corollary 3.4, $b, t \in B$. We immediately see that the regular partitions of the poset $P \oplus Q$ can be obtained in two possible ways only:

(a) by partitioning the posets P and Q independently,

(b) by partitioning the posets P and Q independently, and then making the union of the block of P containing t and the block of Q containing b.

We can thus establish bijections among regular partitions of $P \oplus Q$ of type (a), regular partitions of $P \oplus Q$ of type (b), and pairs (π', π'') where π' is a regular partition of P and π'' is a regular partition of Q. These bijections yield a bijection $\varphi = \Pi_r(P \oplus Q) \to \Pi_r(P) \times \Pi_r(Q) \times C_2$, as follows. Let $C_2 = \{a, b\}$, with $a < b$, and let π be a regular partition of $P \oplus Q$. Clearly, the restriction of π to the elements of P is a regular partition π' of P, and the restriction of π to the elements of Q is the regular partition π'' of Q. Then the bijection φ associates with π the triplet $\varphi(\pi) = (\pi', \pi'', x) \in \Pi_r(P) \times \Pi_r(Q) \times C_2$ such that

- $x = a$ if π is of type (a);
- $x = b$ if π is of type (b).

We next prove that φ preserves the lattice operations. Let $\pi, \sigma \in \Pi_r(P \oplus Q)$, and let $\varphi(\pi) = (\pi', \pi'', x)$ and $\varphi(\sigma) = (\sigma', \sigma'', y)$. By Corollary 4.7, we obtain that $\pi \leqslant \sigma$ in $\Pi_r(P \oplus Q)$ if and only if

- $\pi' \leqslant \sigma'$ in $\Pi_r(P)$, and
- $\pi'' \leqslant \sigma''$ in $\Pi_r(Q)$, and
- $x \leqslant y$ in C_2,

that is, if and only if $(\pi', \pi'', x) \leqslant (\sigma', \sigma'', y)$. Thus, φ is a lattice isomorphism between $P \oplus Q$ and $\Pi_r(P) \times \Pi_r(Q) \times C_2$.

The proof for monotone partitions is similar. □

Example 6.6. Some regular partition lattices of chains are shown in Figure 6.5. We know from Section 6.2 that $\Pi_r(C_n) \cong \mathcal{B}_{n-1}$. Proposition 6.3 provides another proof of this isomorphism. In fact, since $C_2 = C_1 \oplus C_1$, $C_3 = C_2 \oplus C_1$, $C_4 = C_3 \oplus C_1$, and so on, we have $\Pi_r(C_2) \cong C_2$, $\Pi_r(C_3) \cong C_2 \times C_2$, $\Pi_r(C_4) \cong C_2 \times C_2 \times C_2, \ldots$.

6.4 A case of counting

In this section we tackle a more complex instance of enumeration problem. We will provide a formula for regular partitions of the family $\{M_i\}_{i \geqslant 1}$ of partially ordered sets depicted in Figure 6.7.

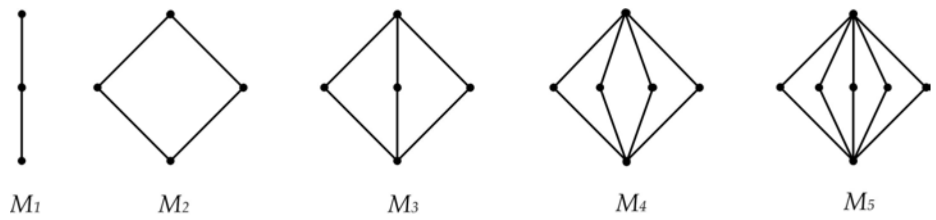


Figure 6.7: *A family of posets.*

Clearly, Proposition 6.3 cannot help us, because even if $M_i = A_1 \oplus A_i \oplus A_1$, the antichain A_i does not have bottom and top, except for the case $i = 1$.

We indicate by t the top of M_i, by r its bottom, and by $m_1, \ldots, m_i$ all the other elements. We can distinguish two kinds of partitions of the underlying set of M_i:

(a) partitions where t and r belong to different blocks, and

(b) partitions where t and r belong to the same block.

It is an exercise to check that each partition of type (a) admits an extension to a regular partition of M_i. Among the partition of type (b) there is the partition $\bar{\pi}$ in a single block. Clearly, such a partition admits an extension to a regular partition of M_i. Let now $\pi \neq \bar{\pi}$ be a partition of type (b), and let m_k be an element of M_i which does not belong to the same block of t in π. Since r and t belong to the same block, we infer $r \lesssim_\pi m_k \lesssim_\pi t \lesssim_\pi r$. By Corollary 3.4, π does not admit an extension to a regular partition of M_i.

We know that the number of partitions of the underlying set of M_i is B_{i+2}. Further, we observe that the number of partitions of type (b) is B_{i+1}, because such partitions can be thought of as partitions of M_i where r and t are identified into a single element. Since all the partitions of type (a) admit an extension to a regular partition of M_i, and all the partitions of type (b), except the single-block one, do not, we have:

$$|\Pi_r(M_i)| = B_{i+2} - B_{i+1} + 1\,.$$

The following table shows the number of regular partitions of M_i for $i \leqslant 10$.

i	1	2	3	4	5	6	7	8	9	10
$\lvert\Pi_r(M_i)\rvert$	4	11	38	152	675	3264	17008	94829	562596	3535028

Example 6.7. We consider the poset M_3 depicted in Figure 6.8. Figure 6.9 shows all the regular partitions of M_3.

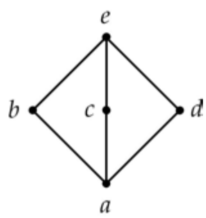


Figure 6.8: *The poset M_3.*

6.5 Counting order-preserving maps

Another problem related to our work is the enumeration of order-preserving maps. As we will see in this section, the study of monotone and regular partitions of a poset can help solving this kind of problems.

We take a step back to the classical problem of counting functions between sets. We know that if A and B are sets, and $|A| = n$, $|B| = x$, the number of functions from A to B is

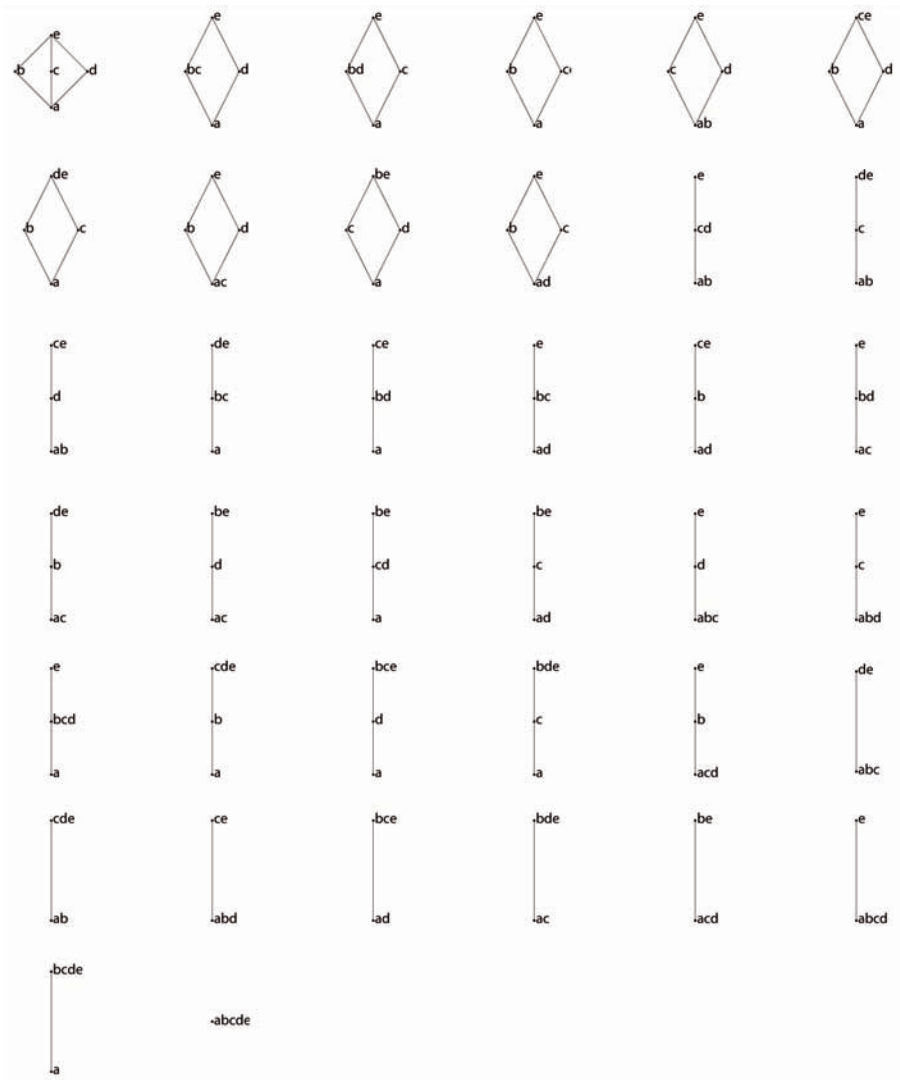


Figure 6.9: *Partitions of M_3.*

x^n, and the number of injections from A to B is $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$. Given the formula to count the number of injective functions, we can also obtain a formula for general functions using the partition lattice of the domain A. In fact, observe that the lattice Π_n is ranked. The rank function on Π_n is $n-k$, where k is the number of blocks of a partition. The Whitney numbers of the second kind of Π_n coincide with the Stirling numbers of the second kind:

$$W_k = S(n, n-k).$$

Then, we note that each function factorizes uniquely as a surjection followed by an injection, and deduce that we can count functions from A to B by counting all the possible injections of all partitions of A into B. We obtain that the number of functions from A to B is

$$\sum_{k=0}^{n} S(n,k)(x)_k = x^n .$$

Remark 6.3. We recall that the Stirling numbers of the second kind satisfy the basic recurrence:

$$S(n,k) = kS(n-1,k) + S(n-1,k-1) .$$

The following table shows the Stirling numbers of the second kind $S(n,k)$ for $n \leqslant 9$.

n	$S(n,1)$	$S(n,2)$	$S(n,3)$	$S(n,4)$	$S(n,5)$	$S(n,6)$	$S(n,7)$	$S(n,8)$	$S(n,9)$
1	1								
2	1	1							
3	1	3	1						
4	1	7	6	1					
5	1	15	25	10	1				
6	1	31	90	65	15	1			
7	1	63	301	350	140	21	1		
8	1	127	966	1701	1050	266	28	1	
9	1	255	3025	7770	6951	2646	462	36	1

We now go back to posets. In `Poset` – see Chapter 2 – we have two dual factorization system. We choose to think of each order-preserving map as a regular epimorphism followed by a monomorphism. Thus, we count order-preserving maps from a poset P to a poset Q by counting all the possible *order-preserving injections* from all the *regular partitions* of P into Q.

We consider the case studied in the previous section and we count the number of order-preserving maps from a poset M_n to a poset M_x. To reach our goal, we need to investigate a bit more deeply the structure of the regular partition lattice $\Pi_r(M_n)$. We know that the total number of regular partitions is given by $B_{n+2} - B_{n+1} + 1$. One can verify at some labor – recalling that M_n has $n+2$ elements – that the Whitney numbers of the second kind $W_0, \ldots, W_{n+1}$ of $\Pi_r(M_n)$ are given by:

$$W_k = \begin{cases} S(n+2, n+2-k) - S(n+1, n+2-k) & \text{if } k \neq n+1 , \\ S(n+2, n+2-k) - S(n+1, n+2-k) + 1 = 1 & \text{otherwise} . \end{cases}$$

For convenience, we denote the k-th Whitney number of the second kind of $\Pi_r(M_n)$, for $k = 0, \ldots, n+1$ by $R(n+2, n+2-k)$. We thus have:

$$R(h,j) = \begin{cases} S(h,j) - S(h-1,j) & \text{if } j \neq 1 , \\ 1 & \text{if } j = 1 . \end{cases}$$

The value $R(h,j)$ represents the $(h-j)$-th Whitney number of the second kind of the lattice $\Pi_r(M_{h-2})$. [1]

h	$R(h,1)$	$R(h,2)$	$R(h,3)$	$R(h,4)$	$R(h,5)$	$R(h,6)$	$R(h,7)$	$R(h,8)$	$R(h,9)$
3	1	2	1						
4	1	4	5	1					
5	1	8	19	9	1				
6	1	16	65	55	14	1			
7	1	32	211	285	125	20	1		
8	1	64	665	1351	910	245	27	1	
9	1	128	2059	6069	5901	2380	434	35	1

As we have seen before, in the classical case of sets and functions the knowledge of these numbers, together with a formula to compute the number of injections, suffices to derive a formula to compute the number of functions. Unfortunately, in the category `Poset` this is not the case in general. In fact, unlike in the set-theoretic case, in a regular partition lattice it is not always true that all the elements having the same rank – that is, all the regular partitions having the same number of blocks – are isomorphic as posets. An example is given in Figure 4.5. However, in the special case at hand all the regular partitions of the same rank are isomorphic. In detail, it is possible to check that:

- the regular partition of M_n with rank $n+1$ is isomorphic to C_1,
- the regular partitions of M_n with rank n are isomorphic to C_2, and
- the regular partitions of M_n with rank $k = 0,\dots,n-1$ are isomorphic to M_{n-k}.

For $n = 3$, see Figure 6.9. What we know now about the regular partition lattice of M_n is sufficient to our aim.

We need now to find a formula to count order-preserving injections from M_n to M_x. Let r be the bottom of M_n, r' be the bottom of M_x, t be the top of M_n, t' be the top of M_x, and let $m_1,\dots,m_n$ be the remaining elements of M_n and $m'_1,\dots,m'_x$ be the remaining elements of M_x. We can easily verify that an order-preserving injection from M_n to M_x must map r into r', t into t', and each element m_i into an element m'_j, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant x$. In other words, the number I_{M_n,M_x} of order-preserving injection from M_n to M_x equals the number of injections from an n element set to an x element set, that is

$$I_{M_n,M_x} = (x)_n\,. \tag{6.1}$$

Since in the regular partition lattice of M_n we also have partitions isomorphic to C_1 and C_2, we still need formulas to count the number I_{C_2,M_x} of order-preserving injections from C_2 to M_x, and the number I_{C_1,M_x} of order-preserving injections from C_1 to M_x. We immediately obtain:

$$I_{C_1,M_x} = x+2\,, \tag{6.2}$$

[1] Incidentally, we notice that the j-th column of the above table is generated by $(\prod_{k=2}^{j}(1-kx))^{-1}$ - see the bibliographic notes.

$$I_{C_2,M_x} = 2x+1\,. \tag{6.3}$$

We are finally ready to derive a formula to count the total number $F_{n,x}$ of order-preserving maps from M_n to M_x. Indeed,

$$\begin{aligned}
F_{n,x} &= W_{n+1}I_{C_1,M_x} + W_nI_{C_2,M_x} + W_{n-1}I_{M_1,M_x} + \cdots + W_1I_{M_{n-1},M_x} + W_nI_{M_n,M_x} = \\
&= R(n+2,1)I_{C_1,M_x} + R(n+2,2)I_{C_2,M_x} + R(n+2,3)I_{M_1,M_x} + \cdots + R(n+2,n+2)I_{M_n,M_x} = \\
&= R(n+2,1)I_{C_1,M_x} + R(n+2,2)I_{C_2,M_x} + \sum_{k=1}^{n} R(n+2,k+2)I_{M_k,M_x}
\end{aligned}$$

Summing up, noting that $R(n+2,2) = 2^n$, from (6.1 - 6.3) we have

$$F_{n,x} = x+2+2^n(2x+1) + \sum_{k=1}^{n} R(n+2,k+2)(x)_k. \tag{6.4}$$

Example 6.8. We want to enumerate order preserving maps from M_3 to M_4. Formula (6.4) gives the answer

$$F_{3,4} = 4+2+2^3(2\cdot 4+1) + \sum_{k=1}^{3}[S(3+2,k+2) - S(3+1,k+2)](4)_k = 286\,.$$

We give a combinatorial interpretation of this formula by counting in how many ways we can inject all the regular partitions of M_3 into M_4. Figure 6.9 tells us that in the regular partition lattice of M_3 there are

- 1 copy of C_1,
- 8 copies of C_2,
- 19 copies of M_1,
- 9 copies of M_2, and
- 1 copy of M_3.

We observe now that there are

- 6 order-preserving injections of C_1 into M_4,
- 9 order-preserving injections of C_2 into M_4,
- $(4)_1 = 4$ order-preserving injections of M_1 into M_4,
- $(4)_2 = 12$ order-preserving injections of M_2 into M_4,
- $(4)_3 = 24$ order-preserving injections of M_3 into M_4.

Thus, the number of order-preserving maps from M_3 to M_4 is

$$F_{3,4} = 1\cdot 6 + 8\cdot 9 + 19\cdot 4 + 9\cdot 12 + 1\cdot 24 = 286\,.$$

6.6 Remarks on Möbius inversion

Let $\{p_n(x)\}_{n\geqslant 0}$ and $\{q_n(x)\}_{n\geqslant 0}$ be two polynomial sequences such that, for all n, the degree of $p_n(x)$ and $q_n(x)$ is n. By elementary linear algebra, there exist two sequences of *connection* *constants* $\{c_{n,k}\}$ and $\{d_{n,k}\}$ such that, for all n,

$$p_n(x) = \sum_k c_{n,k} q_k(x) \quad \text{and} \quad q_n(x) = \sum_k d_{n,k} p_k(x).$$

If we are able to provide a direct counting argument for one of these connection formulas, we can try to find the other formula, the *inverse formula*, by the well-known method of *Möbius inversion*, applied on a suitable poset. In many cases, according to G.-C. Rota et al. [JRS81], *"the most difficult part of this program is to guess the right poset"*. Formally, the *Möbius inversion theorem* states what follows.

Theorem 6.4. *Let P be a given poset with bottom element $\mathbf{0}$, and let f and g be maps from P to a field F such that for all $\sigma \in P$*

$$g(\sigma) = \sum_{\sigma \leqslant \pi} f(\pi). \tag{6.5}$$

Then there exists a unique function $\mu : P \to F$, called Möbius function*, such that*

$$f(\mathbf{0}) = \sum_{\pi \in P} \mu(\pi) g(\pi). \tag{6.6}$$

Computation of the Möbius function for the finite lattices we are going to deal with is quite simple. In fact, we can easily do it recursively. Let L be a lattice, and let $\mathbf{0}$ be the bottom of L. We have $\mu(\mathbf{0}) = 1$, and $\mu(\pi) = -\sum_{\sigma < \pi} \mu(\sigma)$, for each $\pi \in L$, $\pi \neq \mathbf{0}$.

Consider the sequence $\{x^n\}$, which counts the number of functions from a set with n elements to a set with x elements. As already mentioned,

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k. \tag{6.7}$$

The Stirling numbers of the second kind $S(n,k)$ are the Whitney numbers of the second kind of the lattice Π_n. Clearly, we can also write

$$x^n = \sum_{\pi \in \Pi_n} (x)_{\nu(\pi)}, \tag{6.8}$$

where $\nu(\pi)$ is the number of blocks of the partition π. We want to derive the inverse formula of (6.7) by using Möbius inversion on Π_n. We set

$$f(\pi) = (x)_{\nu(\pi)} \quad \text{and} \quad g(\pi) = x^{\nu(\pi)},$$

so that we can rewrite the formula (6.8) as

$$g(\mathbf{0}) = \sum_{\mathbf{0} \leqslant \pi} f(\pi), \tag{6.9}$$

where $\mathbf{0}$ is the bottom of Π_n, that is the partition into singletons. Since for all $\sigma \in \Pi_n$ the poset $P_\sigma = \{\pi \in \Pi_n \mid \sigma \leqslant \pi\}$ is isomorphic to $\Pi_{\nu(\sigma)}$, we can generalize the formula (6.9) to obtain

$$g(\sigma) = \sum_{\sigma \leqslant \pi} f(\pi).$$

Finally, by the Möbius inversion theorem we have

$$(x)_n = f(\mathbf{0}) = \sum_{\pi \in \Pi_n} \mu(\pi) g(\pi) = \sum_{\pi \in \Pi_n} \mu(\pi) x^{\nu(\pi)} = \sum_{k=0}^{n} x^k \sum_{\substack{\pi \in \Pi_n \\ \nu(\pi)=k}} \mu(\pi). \tag{6.10}$$

For ranked posets, we can define the *Whitney numbers of the first kind* by

$$w_k(P) = \sum_{\substack{\pi \in P \\ r(\pi)=k}} \mu(\pi),$$

where $r(\pi)$ is the rank of π. For Π_n the Whitney numbers of the first kind are the Stirling numbers of the first kind $s(n,k)$. Formula (6.10) thus becomes

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k,$$

the inverse of (6.7).

This procedure extends to posets. In this case the *right* posets to use for Möbius inversion are the monotone and regular partition lattices. Specifically,

- if we are able to provide a formula for order-preserving maps from a poset P to a poset Q in terms of the numbers of order-preserving injections, we can do the inversion on the lattice $\Pi_r(P)$ and derive a formula to count order-preserving injections from P to Q in terms of the numbers of order-preserving maps;

- if we are able to provide a formula for order-preserving maps from a poset P to a poset Q in terms of the numbers of order-embeddings (regular monomorphism), we can do the inversion on the lattice $\Pi_m(P)$ and derive a formula to count order-embeddings from P to Q in terms of the numbers of order-preserving maps.

We give here just a numerical example of how this inversion works. What is interesting in this example is that not all the partitions having the same rank are isomorphic. This fact will force us to analyze the structure of Π_r "element by element". To give the following results, we make use of a special Mathematica® package for treating posets and partitions, developed by the author.

We consider the posets P and Q depicted in Figure 6.10. Figure 6.11 shows all the regular partitions of P. Figure 6.12 shows the regular partition lattice of P.

We observe that the lattice Π_r contains

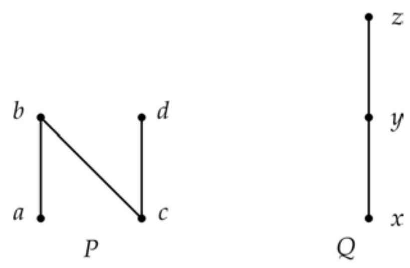


Figure 6.10: *The posets P and Q.*

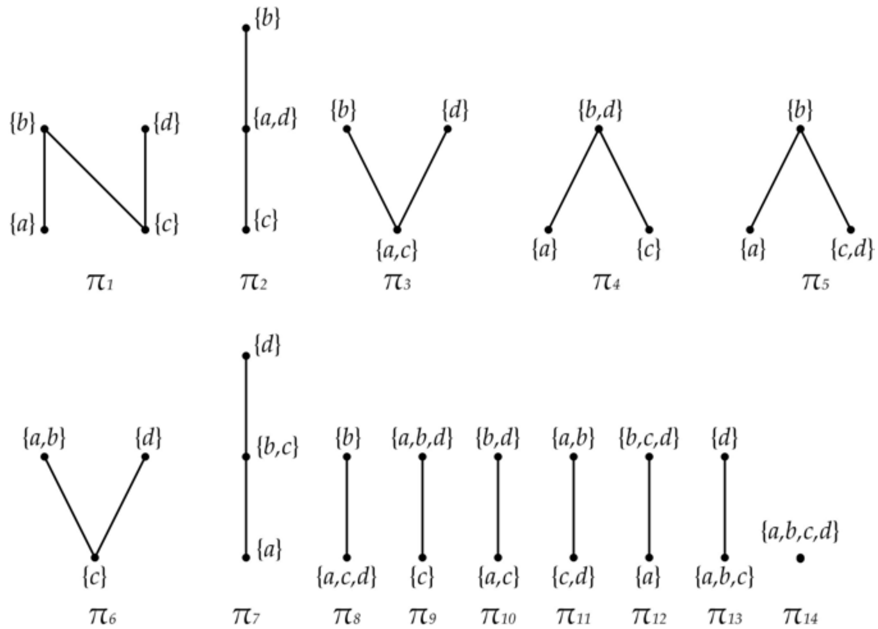


Figure 6.11: *Regular partitions of P.*

- 1 copy of π_1,
- 2 isomorphic copies of π_2,
- 2 isomorphic copies of π_3,
- 2 isomorphic copies of π_4,
- 6 isomorphic copies of π_8, and
- 1 copy of π_{14}.

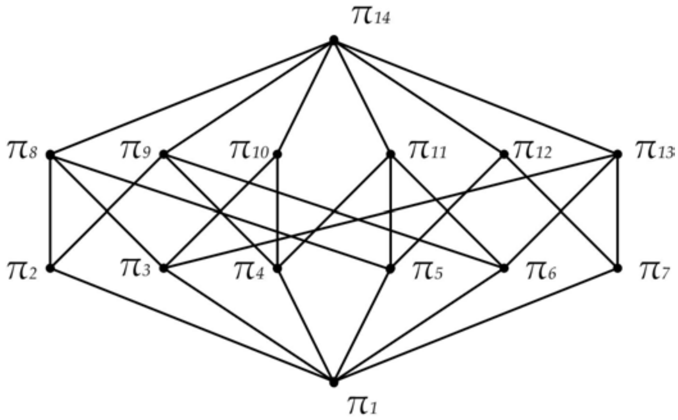


Figure 6.12: *Regular partition lattice of P.*

We first count order-preserving maps from P to Q directly, summing the numbers of order-preserving injections from π_i to Q, for $i = 1, \ldots, 14$. From now on, we indicate by F_i and I_i the number of order-preserving maps and order-preserving injections from π_i to Q, respectively. We have

$$I_1 = 0\,,\ I_2 = 1\,,\ I_3 = 2\,,\ I_4 = 2\,,\ I_8 = 3\,,\ \text{ and }\ I_{14} = 3\,.$$

Summing up, we obtain the total number F_P of order-preserving maps from P to Q:

$$F_P = 1\cdot 0 + 2\cdot 1 + 2\cdot 2 + 2\cdot 2 + 6\cdot 3 + 1\cdot 3 = 31\,.$$

We are now ready to do... the inverse of what we have done. Following the guidelines of the Möbius inversion method, we compute the number I_P of order-preserving injections from P to Q, using the numbers of order-preserving maps from all the regular partitions of P to Q. (Note that we already know our result, because obviously $I_P = I_1 = 0$.) We immediately obtain

$$F_1 = 31\,,\ F_2 = 10\,,\ F_3 = 14\,,\ F_4 = 14\,,\ F_8 = 6\,,\ \text{ and }\ F_{14} = 3\,.$$

From Figure 6.12 we see that the values of the Möbius function for the elements of $\Pi_r(P)$ are

- $\mu(\pi_1) = 1$,
- $\mu(\pi_2) = \mu(\pi_3) = \mu(\pi_4) = \mu(\pi_5) = \mu(\pi_6) = \mu(\pi_7) = -1$,
- $\mu(\pi_8) = \mu(\pi_9) = \mu(\pi_{11}) = \mu(\pi_{13}) = 2$,
- $\mu(\pi_{10}) = \mu(\pi_{12}) = 1$, and
- $\mu(\pi_{14}) = -5$.

Finally, we can compute I_P, obtaining

$$\begin{aligned} I_P &= \mu(\pi_1)I_1 + [\mu(\pi_2)+\mu(\pi_7)]I_2 + [\mu(\pi_3)+\mu(\pi_6)]I_3 + [\mu(\pi_4)+\mu(\pi_5)]I_4 + \\ &\quad +[\mu(\pi_8)+\mu(\pi_9)+\mu(\pi_{10})+\mu(\pi_{11})+\mu(\pi_{12})+\mu(\pi_{13})]I_8 + \mu(\pi_{14})I_{14} = \\ &= 1\cdot 31 - 2\cdot 10 - 2\cdot 14 - 2\cdot 14 + 10\cdot 6 - 5\cdot 3 = 0, \end{aligned}$$

as desired.

6.7 Bibliographic notes

As general references to enumeration problems, we cite the classical books [Sta97], [Aig79], and [Com74]. For more specific problems we mention here [BM02] and [ES91], where the problem of counting finite posets is treated, and [Rot64a], where a discussion on set-partitions can be found. The latter works can be seen as specific cases of our more general problem of counting monotone and regular partitions of posets. More closely related to our work are, for instance, [BM05], [Pfe04], and [Ric98], that study the enumeration of quasiorders. For what concerns the number of order-preserving maps, we cite [Far95] and [DRSW92], where some particular counting problems are tackled.

A very useful general reference for enumeration problems is the website

`http://www.research.att.com/~njas/sequences/`

This page collects a huge number of numerical sequences, with formulas to compute these numbers, when available, links, and references. Sequences related to this chapter are, for instance, `A001035`, which counts the number of posets with n labeled elements, `A000798`, which counts the number of different quasiorders with n labeled elements, and the sequence `A000110` of Bell numbers. One can also find the sequences `A001047` for $R(h,3)$, `A016269` for $R(h,4)$, and `A025211` for $R(h,5)$.

The last part of the chapter is devoted to the Möbius inversion. References on this topic can be found in [BS97], [BP95], [BBR82] and [Rot64b], but we refer in particular to [JRS81]. In this paper the authors apply the Möbius inversion to three specific cases. The first case, involving the polynomial sequences x^n and $(x)_n$, is reported here in Section 6.6. In the introduction, the authors emphasize the role of the choice of the poset on which the inversion takes place.

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