Products in the category of forests and p-morphisms via Delannoy paths on Cartesian products

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In [5], the authors introduce a technique to compute finite coproducts of finite Gödel algebras, i.e. Heyting algebras satisfying the prelinearity axiom \((\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)\). To do so, they investigate the product in the category opposite to finite Gödel algebras: the category of forests and open order-preserving maps, alias p-morphisms, which we denote by \(F\). (A forest is a partially ordered set \(F\) such that, for every \(x \in F\), the set of lower bounds of \(x\) forms a chain, when endowed with the order inherited from \(F\).) To achieve their result, the authors make use of ordered partitions of finite sets and of a specific operation — called merged-shuffle — on ordered partitions. In [1, Section 4.2], the authors present an alternative, recursive construction of finite products in the category of forests and open order-preserving maps.

In the present work we introduce a further construction of the same finite products, based on products of posets along with a generalization of the combinatorial notion of Delannoy path. The new and most interesting aspect of our construction is that, dually, it uncovers a key relationship between the co-products of finite Gödel algebras and the coproducts in the category of finite distributive lattices. Our main result explains the former coproducts in terms of a construction on the latter; the construction itself is currently best understood via duality using a generalisation of the Delannoy paths.

Classically, a Delannoy path (see [4, p.80]) is a path on the first integer quadrant \(\mathbb{N}^2 \subseteq \mathbb{Z}^2\) that starts from the origin and only uses northward, eastward, and north-eastward steps. We begin by generalizing the notion of Delannoy path to Cartesian products of finite posets. A (finite) path on a poset \(P\) is a sequence \(< p_1, p_2, \ldots, p_n >\) of elements of \(P\) such that \(p_i < p_j\) whenever \(i < j\). (A path on \(P\) is therefore the same thing as a chain of \(P\).) For each \(i \in \{1, \ldots, n-1\}\), the pair \(p_i, p_{i+1}\) is called a step of the path. Given a poset \(P\), and two elements \(p, q \in P\), we write \(p < q\) to indicate that \(q\) covers \(p\) in \(P\), that is, \(p < q\) and for every \(s \in P\), if \(p \leq s \leq q\), then either \(s = p\) or \(s = q\).

In [3], the notion of Delannoy path has been extended to finite products of chains. The following generalization is perhaps less obvious.
Definition 1. Let $P_1, P_2, \ldots, P_n$ be posets, and let $P = P_1 \times P_2 \times \cdots \times P_n$ be their (Cartesian) product. Let $< p_1, p_2, \ldots, p_h >$ be a path on $P$. The step from $p_i = (p_{i,1}, p_{i,2}, \ldots, p_{i,n})$ to $p_{i+1} = (p_{i+1,1}, p_{i+1,2}, \ldots, p_{i+1,n})$ is a Delannoy step, written $p_i \prec p_{i+1}$, if and only if there exists $k \in \{1, \ldots, n\}$ such that $p_{i,k} \neq p_{i+1,k}$, and for each $j \in \{1, \ldots, n\}$, $p_{i,j} = p_{i+1,j}$, or $p_{i,j} < p_{i+1,j}$. The path $< p_1, p_2, \ldots, p_h >$ on $P$ is a Delannoy path if and only if $p_1$ is a minimal element of $P$, and for each $i \in \{1, \ldots, n-1\}$, $p_i \prec p_{i+1}$.

A Delannoy path on $P$ is thus a sequence of Delannoy steps starting from a minimal element of $P$. Delannoy paths on a poset $P = P_1 \times \cdots \times P_n$ can be partially ordered by $< q_1, \ldots, q_m > \leq < p_1, \ldots, p_h >$ if and only if $m \leq h$ and $q_i = p_i$ for each $i \in \{1, \ldots, m\}$. We denote by $D(P_1, \ldots, P_n)$ the poset of all Delannoy paths on $P$. Clearly, $D(P_1, \ldots, P_n)$ is a forest.

Definition 2 (Product). Let $F$ and $G$ be forests. We call $F \times_F G = D(F,G)$ the product of $F$ and $G$.

Definition 3 (Projections). Let $F$ and $G$ be forests, let $\{f_1, \ldots, f_m\}$ and $\{g_1, \ldots, g_n\}$ be the underlying sets of $F$ and $G$, respectively, and let $D = F \times_F G$. We define a function $\pi_F : D \rightarrow F$ such that for each Delannoy path $d \in D$, with $d = \langle (f_i, g_j), \ldots, (f_h, g_k) \rangle$, $\pi_F(d) = f_h$. Analogously, we define a function $\pi_G : D \rightarrow G$ such that $\pi_G(d) = g_k$.

Our main result follows.

Theorem 1. Let $F$ and $G$ be forests. Then

$$F \overset{\pi_F}{\rightarrow} F \times_F G \overset{\pi_G}{\rightarrow} G$$

is the product of $F$ and $G$ in the category $F$.

Remark. We point out the parallel with [2], where the authors use the forest of all paths on a finite poset to construct the Gödel algebra freely generated by a finite distributive lattice.

References