PROFINITE HEYTING ALGEBRAS, AND PARTITIONS OF IMAGE-FINITE POSETS UNDER OPEN MAPS

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1. Introduction, and motivation

We are interested in developing a theory of quotient objects (i.e. partitions) in sufficiently general categories related to non-classical propositional logics. By way of motivation, consider the duality between the category of finite sets and functions, $\mathtt{Set_f}$, and the category of finite Boolean algebras and their homomorphisms, $\mathtt{Boole_f}$. In the former, a quotient object is (the equivalence class of) an epimorphism (=surjection) $f\colon A \twoheadrightarrow B$. Traditionally, one does away with the arrow f, and directly defines a partition of A as a collection of nonempty subsets of A that are mutually disjoint and cover A; that is, one characterises the set $\{f^{-1}(b) \mid b \in B\}$ of fibres of f. (As a third way to look at quotient objects, one shows that partitions of A are in natural bijection with the equivalence relations on A, but here we shall not deal with this important point of view.) Dualising to $\mathtt{Boole_f}$, one has the dual correspondence between (equivalence classes of) monomorphisms $f^*\colon 2^B \rightarrowtail 2^A$, and subalgebras of 2^A. As a first interesting case study, in this paper we obtain the appropriate notion of partition for the objects of the category that is dual to profinite Heyting algebras.

A Heyting algebra is *profinite* if it is representable as an inverse limit of an inverse family of finite Heyting algebras. Let `ProHA` be the category of profinite Heyting algebras and their complete homomorphisms, i.e. homomorphisms preserving arbitrary joins and meets. Given a poset P (with partial order denoted by $\leq$) and a subset $S \subseteq P$, write $\downarrow S = \{x \in P \mid x \leq y, \text{ for some } y \in S\}$ for the *lower set* generated by S, and say S *is* a lower set if $\downarrow S = S$. Upper sets are defined analogously using $\uparrow$. Call a poset P *image-finite* if $\downarrow S$ is finite whenever $S \subseteq P$ is finite. Let us recall that an order-preserving function $f\colon P \to Q$ between posets is called *open* if whenever $f(u) \geq v'$ for $u \in P$ and $v' \in Q$, there is $v \in P$ such that $u \geq v$ and $f(v) = v'$. Open maps are also known as *bounded morphisms*, or *p-morphisms*. Let `ImOPos` be the category of image-finite posets and open maps.

In [BB08], Guram and Nick Bezhanishvili prove the following

Theorem. `ProHA` is dually equivalent to `ImOPos`.

The functor from posets to Heyting algebras implementing one half of this duality is described as follows. If P is any poset, the family $\mathrm{Sub}\, P$ of all lower sets of P partially ordered by inclusion is a complete distributive lattice, and thus carries a unique Heyting implication adjoint to the lattice meet operation. Explicitly,

$$x \to y = \bigvee \{z \in P \mid z \wedge x \leq y\}$$

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for all $x, y \in \mathrm{Sub}\, P$. If, moreover, P is image-finite, $\mathrm{Sub}\, P$ happens to be a profinite Heyting algebra. As to morphisms, given an open order-preserving map $f\colon P \to Q$, the function $\mathrm{Sub}\, f\colon \mathrm{Sub}\, Q \to \mathrm{Sub}\, P$ given by $Q' \mapsto f^{-1}(Q')$ is a complete homomorphism of Heyting algebras. Here we omit the description of the functor from profinite Heyting algebras to image-finite posets that features in the theorem above.

Our objective, then, is to obtain a notion of partition for an image-finite poset P that (i) dualises subobjects in `ProHA`, and (ii) can be directly described in terms of 'parts' of P, without reference to epic arrows in `ImOPos`. This we accomplish in the next section.

2. Subalgebras and open partitions

Consider again the category `ImOPos` of image-finite posets and open maps. In the light of the foregoing discussion, we would like to define partitions of an image-finite poset in terms of fibres of open "surjections". In categories where there is no non-trivial hierarchy of epimorphisms, we may safely let epimorphisms play the rôle that surjections play in $\mathtt{Set}_f$. In the case at hand, we observe that the category `ImOPos` has an (epi , mono) factorisation system, that is, each morphism factorises as an epimorphism followed by a monomorphism, and this factorisation is essentially unique. Moreover, there is no other factorisation system in `ImOPos`. Further, in `ImOPos` epimorphisms are precisely surjective open maps. Finally, surjective open maps precisely correspond to monomorphisms in `ProHA`, hence to complete subalgebras of profinite Heyting algebras. We thus define as follows.

Definition 2.1. An *open partition* of an image-finite poset P is the set-theoretic partition $\pi = \{B_q \mid q \in Q\}$ of P that is induced by some surjective open map $f\colon P \to Q$ onto a poset Q. That is, for each $q \in Q$,

$$B_q = f^{-1}(q) = \{p \in P \mid f(p) = q\}\ .$$

Remark 1. With this definition, a partition of P is not *prima facie* an object of `ImOPos`. However, any open partition π of P carries a canonical underlying order – namely, define

$$B_q \preceq B_u$$

if and only if

$$q \le u \ \text{ in } \ Q\ .$$

Obviously π ordered in this manner is an isomorphic copy of Q.

Our main result is that, just like in $\mathtt{Set}_f$, in `ImOPos` it is possible to do away with the epimorphism f, and define partitions of image-finite posets in intrinsic terms.

Theorem 2.2. *Let P be an image-finite poset, and let $\pi = \{B_i \mid i \in I\}$ be a set-theoretic partition of P, where I is some index set. Then π is an open partition of P if and only if for each $B_i \in \pi$ there exists a subset $J \subseteq I$ such that*

$$\uparrow B_i = \bigcup_{j \in J} B_j\ .$$

In this case, the underlying order $\preceq$ of π is uniquely determined by

$$B_i \preceq B_j \ \text{ iff } \ B_j \subseteq \uparrow B_i \ \text{ iff there are } \ x \in B_i, y \in B_j \text{ with } x \le y\,,$$

for each $B_i, B_j \in \pi$.

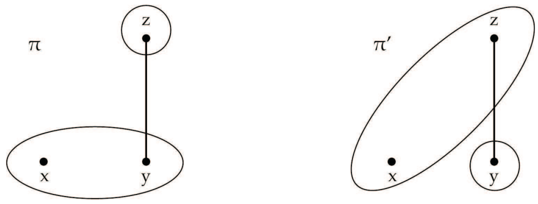

FIGURE 1. Two set-theoretic partitions of a poset.

Example 1. We consider two different set-theoretic partitions $\pi = \{\{x, y\}, \{z\}\}$, and $\pi' = \{\{x, z\}, \{y\}\}$ of the same poset P. The partitions are depicted in Figure 1. It is immediate to check, using Theorem 2.2 or by direct inspection, that π is an open partition of P, while π' is not.

We conclude with some general remarks. While arrows in the category `ImOPos` may only be factorised in one way, namely, through the (epi , mono) factorisation system, this fails in the category of posets and order-preserving maps. Here, both (regular epi , mono) and (epi , regular mono) factorisation systems exist, and the classes of epimorphisms and regular epimorphisms do not coincide. This is due to the absence of the Heyting implication, that is, of the openness condition on maps. In this case, two different notions of 'partition' will arise by taking fibres of epimorphisms and regular epimorphisms, respectively. For finite distributive lattices and their dual posets (though not for Heyting algebras), an in-depth combinatorial analysis of such notions of partitions can be found in [Cod08]. Open partitions of finite posets are introduced and applied in [CDM09] to the problem of developing an analogue of probability theory over prelinear intuitionistic logic; this, of course, is a special case of the profinite theory dealt with here.

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