A COMBINATORIAL EXPANSION ARISING FROM ŁUKASIEWICZ LOGIC

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This is a joint work with O. M. D'Antona and V. Marra. In this work, we consider possible generalisations of the basic expansion $x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$ to multisets arising as Birkhoff duals of finite MV algebras.

Following [1], we consider the category $\mathsf{C}_{\mathrm{fin}}$ whose objects are finite *multisets*, i.e. functions $\alpha\colon A \to \mathbb{N}$, with A a finite set. A morphism $f\colon \alpha \to \gamma$ between multisets $\alpha\colon A \to \mathbb{N}$ and $\gamma\colon C \to \mathbb{N}$ is a function $f\colon A \to C$ such that for all $a \in A$, $\gamma \circ f(a) \mid \alpha(a)$, where $s \mid t$ stands for "s divides t".

One sees [1, Theorem 6.8] that $\mathsf{C}_{\mathrm{fin}}$ is dually equivalent to the category of finite MV-algebras [2]. MV-algebras are to Boolean algebras as Łukasiewicz logic is to Boolean logic. Thus, our multisets generalize sets in the same sense as Łukasiewicz logic generalize classical logic.

To generalize $x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$, we proceed as follows. We associate with a multiset α a partition ν of an integer determined by its multiplicities. We say that α is a *ν-set*. Then, we introduce the notions of *weak* and *strong partitions* of a multiset, and write $\left\{\left\{ {\nu \atop \kappa} \right\}\right\}$ and $\left\{ {\nu \atop \kappa} \right\}$ for the number of *weak κ-partitions* (*i.e.*, partitions which are κ-sets) and *strong κ-partitions* of a ν-set, respectively. We also introduce the notions of *weak subset* and *strong subset* in an appropriate manner, and write $((\chi))_\nu$ and $(\chi)_\nu$ for the number of weak ν-subsets and strong ν-subsets of a χ-set, respectively. Let χ^ν denote the number of functions from a ν-set to a χ-set. The promised generalisations of $x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k$ are obtained as follows.

Theorem. *For any two partitions of integers ν and χ, we have*

$$\sum_\kappa \left\{\left\{ {\nu \atop \kappa} \right\}\right\} (\chi)_\kappa = \chi^\nu = \sum_\kappa \left\{ {\nu \atop \kappa} \right\} ((\chi))_\kappa ,$$

where κ ranges over all partitions of a integer.

[1] R. L. O. Cignoli, E. Dubuc, and D. Mundici, *Extending Stone duality to multisets and locally finite MV-algebras*, J. Pure Appl. Algebra 189, no. 1-3, 37–59, 2004.

[2] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici, *Algebraic foundations of many-valued reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.

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